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Title: MODELING OF COMPOSITE BEAMS AND PLATES FOR STATIC AND DYNAMIC ANALYSIS

PROJECT ADMINISTRATION DATA

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NOTICE OF PROJECT CLOSEOUT

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Final Invoice or Copy of Final Invoice	Y	_____
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Modeling of Composite Beams and Plates for Static and Dynamic Analysis

**Interim Semi-Annual Report
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Introduction

The purpose of this research is to develop a rigorous theory and corresponding computational algorithms for a variety of problems regarding the analysis of composite beams and plates. The modeling approach is intended to be applicable to both static and dynamic analysis of generally anisotropic, nonhomogeneous beams and plates. The major part of the effort during this first reporting period has been devoted to development of a theory for analysis of the local deformation of plates. In addition, some work has been performed on global deformation of beams. Because of the strong parallel between beams and plates, we will treat the two together as *thin bodies*, especially where we believe it will aid in clarifying to the reader the meaning of certain terminology and the motivation behind certain mathematical operations.

Background

The static and dynamic analysis of beams and plates is of fundamental importance in many engineering problems. In design and analysis of modern aerospace systems, subsystems which consist of laminated plates or composite beams are often encountered such as wing, fuselage, and rotor blade structures. Classical beam and plate theories are known to be adequate for many applications. For beams or plates with anisotropic and nonhomogeneous construction, however, these theories suffer from several sources of inaccuracy. For example, composite beams – even those which are slender – can be very flexible in shear. When such beams are analyzed by classical theories, which generally ignore shear deformation, certain kinematical quantities and associated constitutive couplings are absent which are known to be important. Rehfield, Atilgan, and Hodges (1990) concluded that such phenomena can influence the global deformation in thin-walled beams designed for extension-twist coupling by as much as a factor of two! One of the chief mechanisms for this large effect is a coupling between bending and shear deformation due to anisotropic materials (see, Rehfield and Atilgan, 1989).

This conclusion also holds for beams which are not necessarily thin-walled; however, the cross sectional analysis of general nonhomogeneous, anisotropic beams cannot be treated without finite elements. Unlike classical beam or plate theories, theories for composite beams or plates must contain some means of calculating the elastic properties. These properties are not merely material moduli multiplied by certain integrals over the section of the beam. A review by Hodges (1990a) covers most of the literature dealing with modeling of beams prior to 1988. A similar treatment of composite plates, in which the determination of properties is carried out as a separate analysis, appears to be missing from the literature.

The analysis of deformation for composite thin bodies normally must be done as an iterative process. There are local deformation analyses, which determine the elastic constants and details of local deformation in terms of global deformation parameters. Also there are global analyses, which determine the global (beam- or plate-like deformation such as bending, extension, torsion, and shear) deformation. One could proceed as follows: (1) calculate the local deformation and elastic constants for the undeformed structure; (2) use

the resulting elastic constants in a global deformation analysis; (3) recalculate the local deformation for the deformed structure; (4) stop the process if the elastic constants do not change more than some tolerance or else go back to step 2. Most plates and many beams do not exhibit a sufficiently large local deformation (which we will call warping) to require more than one calculation of properties. In other words, elastic constants which are determined based on the undeformed state may be sufficient even for geometrically nonlinear analyses.

There is another aspect of research which we originally proposed, that being the use of space-time finite elements to treat the global dynamics of beams. In the time between the submittal of our proposal to the Aerostructures Directorate and its being funded, another of our proposals was funded. It had been under review at the National Science Foundation for a long time; among other things, it concerns the dynamics of beams by space-time finite elements. Since both of these grants have this common subject matter as subsets of their programs, we intend to treat this portion of the research as jointly funded. Prof. David A. Peters and Dr. Weiyu Zhou have contributed to the NSF work, and thus, indirectly to this project.

Most finite element procedures for time-dependent phenomena are based on semi-discretizations; finite elements are used in space to reduce to a system of ordinary differential equations. This kind of procedure is widely used in practice and fairly well understood. Space-time finite elements have been rarely used in solutions of engineering dynamic problems. Classic time integration methods are usually included in computational procedures. Recent developments of the space-time finite element method allow application of approximation techniques to the spatial and temporal domains. Special schemes lead to highly efficient algorithms that reduce both memory requirements and number of arithmetical operations.

The use of space-time finite elements presents yet another duality. The static and frequency-domain global deformation analyses for plates are solved on the 2-D domain of the plate; the space-time dynamics of beams can be cast as a 2-D problem with simultaneous spatial and temporal discretizations.

Previous Work Related to Beams

Although there has been much work published in the literature on beams and plates, a unified approach such as we are undertaking has not received very much attention. Furthermore, most of what does exist in the literature applies to beams only. Thus, before getting into plate references, we will describe the analogous type of analysis for beams with the hope of clarifying what we have begun to do for plates.

Berdichevsky (1981) was the first to prove, from variational-asymptotic considerations, that nonlinear analysis of beams can be split into two separate problems: the local deformation is a linear 2-D problem, and the global deformation is a nonlinear 1-D problem. The global deformation analysis of beams can be undertaken by use of 6 displacement variables associated with a cross section of a beam. The variables describe displacement of a point in the cross section and rotation of the cross section as a rigid body. There are

also 6 generalized strains (extension and two shearing strains at the reference line, twist, and two bending "curvatures") and 6 stress resultants (axial force and two components of shear force, twisting moment, and two components of bending moment); these must be related by elastic constants. This way of describing the global deformation requires at most 21 elastic constants (a symmetric 6×6 matrix).

Beam Local Deformation To find these constants, all possible local deformations (*i.e.*, warping) of the cross section must be taken into account, although they can be assumed small. Here warping refers not only to out-of-plane distortion of the cross section during torsional deformation as in classical theories, but in-plane and out-of-plane deformation; these are fully coupled when the beam is nonhomogeneous and anisotropic.

As pointed out by Hodges (1990a), the work by Giavotto *et al.* (1983) is the most general of all the published works. Giavotto *et al.* (1983) developed a finite element approach for determining the elastic constants for an arbitrarily nonhomogeneous, anisotropic beam. This approach makes use of a two-dimensional finite element mesh representative of the beam cross section geometry and material properties. The results, in addition to the matrix of elastic constants, include the distribution of warping displacements per unit values of each of the stress resultants (section forces and moments in the cross section basis) and the three-dimensional stress and strain values throughout the cross section per unit values of the stress resultants.

Because of the generality of the work of Giavotto *et al.* (1983), we have adapted a version of the code developed by Giavotto, Borri, and their associates for obtaining the elastic constants for anisotropic beams. Its only shortcoming is its rather long computer times for calculating properties of realistic composite beams. For this reason, under Army Research Office sponsorship, we are also developing a theory based on the variational-asymptotic method as formulated and applied to nonlinear analysis of isotropic shells by Berdichevsky (1978, 1979). This method has some promise of yielding a more computationally efficient algorithm for extracting the properties.

Beam Global Deformation These elastic constants can then be used to find either linear or nonlinear global deformation, free-vibration modes and frequencies (see, Hodges and Atilgan *et al.*, 1989, 1990), and buckling behavior (see, Rehfield and Atilgan, 1989). Various modal, direct numerical integration, and finite element methods exist for this purpose. Because of their computational efficiency and modeling flexibility, finite element methods are quite popular. Displacement finite element methods for geometrically nonlinear behavior of beams, however, require numerical quadrature of highly nonlinear functions of the beam deformation. This tends to make the numerical solution procedure quite inefficient. On the other hand, with a mixed method such as that of Hodges (1990b), numerical element quadrature can be avoided (as long as applied load terms that are explicit in the axial coordinate are integrable in closed form). Such a nonlinear global analysis gives the beam displacements, rotations, extensional strain, shear strains, twist, bending curvatures, and sectional forces and moments to a comparable level of accuracy. It should be noted that

one can then use these results for forces and moments to find pointwise stress or strain levels throughout the cross section using the cross sectional finite element mesh.

Beam behavior in the time domain can be calculated by finite elements. For the dynamics of beams, recent work in the field of space-time finite elements applied to structural dynamics are described and reviewed by Bajer and Bonthoux (1988). There have been several developments in this area for time domain analysis of simple linear oscillators and rigid bladed helicopter rotors; for example Borri *et al.* (1985, 1988), Borri (1986), and Peters and Izadpanah (1988). The results indicate that finite elements in time provide a way of determining the dynamic behavior of a deformable body undergoing time-dependent loading. We have concentrated our work on the determination of the dynamic response of flexible structures by simultaneous discretization of the spatial and temporal domains. The main purpose is to determine if this methodology can be made feasible.

Previous Work Related to Plates

Reissner (1985) departed from a three-dimensional statement of the problem recalling the fact that in the absence of the adjective *thin*, plate theory would be no more than a class of boundary value problems in three-dimensional elasticity. He gave an excellent sociological-historical survey with his interpretations of the nature of approximations and their consequences. He also touched upon the qualitative differences in the modeling of laminated plates. Since we believe that a *consistent* plate theory should include all possible deformations, our starting point is, naturally, the three-dimensional kinematics.

The analysis of laminated plates has attracted an enormous amount of attention. A description of the plate problem, analogous to the beam, has not been published to the best of our knowledge. Although the review articles by Bert (1984) and Noor and Burton (1989) contain hundreds of references which treat the laminated plate problem, all of them use some sort of explicit, analytic, through-the-thickness assumptions for displacement. While this is not incorrect, simple, low-order theories of this type will not yield correct stresses and strains through the thickness. The reason for this is rather obvious: in a laminated plate where each layer has distinct material properties, the actual displacement is not and cannot be analytic. When the assumed displacement is differentiated to yield the strain, the approximate strain will also be analytic while the actual strain may be discontinuous. Higher-order theories will be more accurate, but at a potentially high cost.

There are alternatives. One is to use a family of approaches as outlined and reviewed by Librescu and Reddy (1989) in which breaking the plate into finite elements through the thickness is advocated. This will yield the correct answer, but, again, at a potentially high computational cost. Another alternative is to derive a local deformation theory similar to that for the beam described above.

Present Approach

Plate Modeling In this research, we are developing such a computational method for determining the elastic constants for laminated plates. The approach is very similar to

that described above for beams. Yet, to the best of our knowledge, the present approach has never been attempted. We originally set out to solve the interior St.-Venant solution for the plate by finite elements in a manner that is strictly analogous to the approach of Giavotto *et al.* (1983) for the beam problem. After applying the finite element method, a set of linear equations were obtained that were very similar to those of Giavotto *et al.* (1983). Solution of these equations in a manner similar to the earlier ones, however, turned out to be difficult. Thus, we turned to an approximate solution based on the variational-asymptotic method.

The domain of the local deformation problem for the beam is planar (2-D), just as the global deformation problem for the plate is. On the other hand, the global deformation problem for the beam is solved along a line (1-D), just as the local deformation problem for the plate is. For the local deformation of the plate, instead of an arbitrary interior cross section as with the beam, we work with an arbitrary interior normal line element of the plate (a line of material points normal to the reference surface of the plate). The tractions acting on this line element are used to obtain a variational principle governing the local stress resultants and deformation of the line element. The variational principle leads to a symmetric 8×8 matrix of elastic constants (for a total of 36) based on the linear relation between the 8 stress resultants and 8 generalized strains.

As with the beam problem, the elastic constants will be determined from a finite element code that is linear. This code will enable us to calculate the elastic constants for an arbitrary laminated plate. In this report, after presentation of the theory, we present some preliminary results. These results, which essentially duplicate classical theory, were obtained to check out the methodology and the code.

Dynamics of Beams An important step towards obtaining a general and consistent form of beam elastodynamic equations was taken by Hodges (1990b). Therein, geometrically nonlinear beam elastodynamic equations as derived from Hamilton's Principle; also, using appropriate Lagrange multipliers a mixed variational formulation suitable for space-time finite element discretization was developed. In order to exploit the usefulness of space-time finite elements we decided to start from the very basic linear equations for longitudinal dynamics of a beam, a special case of Hodges (1990b), and their solutions. In this report, after a brief treatment of the theory, there are a few preliminary results presented.

The report closes with a description of work to be undertaken during the next reporting period.

Unified Variational Formulation for Anisotropic Plates

Our starting point is the nonlinear kinematics of deformation for plates. We will develop the three-dimensional Biot strain field based upon the kinematical development of Danielson (1989). After this, we will formulate the strain energy. Finally, approximations of this strain energy function will be discussed and asymptotically correct solutions for the through-the-thickness analysis of isotropic plates, as well as the corresponding elastic constants, will be given.

Kinematics

A plate is a flexible body in which matter is distributed about a planar surface so that one dimension is significantly smaller than the other two. (Much of our analysis can be easily extended to treat shells, but herein we will consider only plates.) The reference surface is an arbitrary planar surface, not necessarily the mid-surface of the plate. Throughout the analysis, Greek indices assume values 1 or 2, Latin indices assume values 1, 2, and 3 and repeated indices are summed over their ranges. Let us establish a Cartesian coordinate system x_i so that x_α denote lengths along orthogonal lines in the reference surface and x_3 is the distance the normal to the reference surface. Let \mathbf{b}_i denote an orthogonal reference triad along the undeformed coordinate lines. The position vector to an arbitrary point along the normal line is

$$\mathbf{r}^*(x_1, x_2, x_3) = \mathbf{r}(x_1, x_2) + x_3 \mathbf{b}_3 = x_i \mathbf{b}_i \quad (1)$$

Covariant and contravariant undeformed base vectors are defined as, respectively,

$$\begin{aligned} \mathbf{g}_i &= \frac{\partial \mathbf{r}^*}{\partial x_i} \\ \mathbf{g}^i &= \frac{1}{2\sqrt{g}} e_{ijk} \frac{\partial \mathbf{r}^*}{\partial x_j} \times \frac{\partial \mathbf{r}^*}{\partial x_k} \end{aligned} \quad (2)$$

where $g = \det(\mathbf{g}_i \cdot \mathbf{g}_j)$. For this analysis, both reduce to

$$\mathbf{g}_i = \mathbf{g}^i = \mathbf{b}_i \quad (3)$$

In a similar manner, consider the deformed state configuration. The particle which had position vector $\mathbf{r}^*(x_1, x_2, x_3)$ in the undeformed plate now has position vector $\mathbf{R}^*(x_1, x_2, x_3)$ relative to the same point, which can be represented by

$$\mathbf{R}^*(x_1, x_2, x_3) = \mathbf{R}(x_1, x_2) + x_3 \mathbf{B}_3(x_1, x_2) + w_i(x_1, x_2, x_3) \mathbf{B}_i(x_1, x_2) \quad (4)$$

where $\mathbf{R}(x_1, x_2) = \mathbf{r}(x_1, x_2) + \mathbf{u}(x_1, x_2)$ and $\mathbf{u} = u_i \mathbf{b}_i$ is the displacement vector of the points on the reference surface and $w_i(x_1, x_2, x_3)$ is the general local (*i.e.*, warping) displacement of an arbitrary point on the normal line, consisting of both in- and out-of-plane components, so that all possible deformations are considered (Figs. 1, 2). The relationship between \mathbf{B}_i and \mathbf{b}_i is given by

$$\mathbf{B}_i(x_1, x_2) = C_{ij}(x_1, x_2) \mathbf{b}_j \quad (5)$$

where $C(x_1, x_2)$ is the matrix of direction cosines. Covariant deformed base vectors

$$\mathbf{G}_i = \frac{\partial \mathbf{R}^*}{\partial x_i} \quad (6)$$

can be obtained by standard means. It should be noted, however, the measure numbers w_i provide redundant information since the normal line undergoes rigid-body displacement due to u_i and rigid-body rotation due to C . Therefore, some means of removing this redundancy must be introduced. For a plate, we can *choose* the unit vectors \mathbf{B}_i so that w_i is small, at least in some sense. Setting an appropriate number of weighted average displacements to zero is one way to remove the redundancy. This can be conveniently done by the finite element method and will be dealt with below.

Here we restrict ourselves to the case when strain and local rotation are small so that the three-dimensional Biot strain can be expressed as

$$\Gamma^* = \frac{A + A^T}{2} - I \quad (7)$$

where $A_{ij} = \mathbf{B}_i \cdot \mathbf{G}_k \mathbf{g}^k \cdot \mathbf{b}_j$ is the deformation gradient matrix and I is the 3×3 identity matrix. Here Γ^* is a 3×3 symmetric matrix. Introducing the column matrix w with components w_i , one can expressed the three-dimensional strain field as a 6×1 column matrix $\Gamma = [\Gamma_{11} \ 2\Gamma_{12} \ \Gamma_{22} \ 2\Gamma_{13} \ 2\Gamma_{23} \ \Gamma_{33}]^T$ so that

$$\Gamma = \mathcal{H}\epsilon + I_3 w_{,3} + I_1 w_{,1} + I_2 w_{,2} \quad (8)$$

where $(\)_{,i}$ denotes the partial differentiation with respect to x_i and

$$\begin{aligned} \mathcal{H} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & x_3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & x_3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & I_3 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ I_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & I_2 &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (9)$$

and $\epsilon = [\gamma \ \kappa]^T$, the intrinsic strain measure which is function of only x_α . Also $\gamma = [\gamma_{11} \ 2\gamma_{12} \ \gamma_{22} \ 2\gamma_{13} \ 2\gamma_{23}]^T$ and $\kappa = [\kappa_{11} \ 2\kappa_{12} \ \kappa_{22}]^T$ are the so-called force and moment strain measures. Note that γ_{11} and γ_{22} are the extensional strains of the reference surface, γ_{12} is the shear strain in the plane of the reference surface, $\gamma_{\alpha 3}$ are the transverse shear strains of the normal line element, κ_{11} and κ_{22} are the elastic components of the bending curvature, and κ_{12} is the elastic twist. The force and moment strain measures are so

designated because they are conjugate to the actual running stress and moment resultants, respectively.

Since warping displacements are supposed to be quite small, the few nonlinear terms in the strain field, which couple w and ϵ , have been neglected in Eq. (8). The form of the strain field is of great importance because it is now linear in γ , κ , and w and its derivatives. If the top and the bottom of the line element through the thickness of the plate are free of tractions, application of the principle of virtual work to an infinitesimal line element, would lead to a system of *linear* equations over the one-dimensional line element governing w . The warping could then be determined in terms of these intrinsic strain measures or stress resultants as in Giavotto *et al.* (1983). This would lead to a unique two-dimensional strain energy function $U(\gamma, \kappa)$. The elastic law could then be put in a form

$$\begin{Bmatrix} F \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix} \quad (10)$$

where F and M are column matrices $F = [F_{11} \ F_{12} \ F_{22} \ F_{13} \ F_{23}]^T$ and $M = [M_{11} \ M_{12} \ M_{22}]^T$. Here F_{11} and F_{22} are the in-plane stretching stress resultants, F_{12} is the in-plane shear stress resultant, $F_{\alpha 3}$, are the transverse shear stress resultants; M_{11} and M_{22} are the bending moment resultants and M_{12} is the twisting moment resultant, all expressed in the B_i basis. The elastic stiffness matrix relating F and M to γ and κ is 8×8 . The matrix A is 5×5 , D is 3×3 , and B is 5×3 .

Even though this methodology is successful for beam formulations (for linear analysis see Giavotto *et al.*, 1983, and for nonlinear analysis see Atılgan and Hodges, 1990), it has not been completely resolved for plate analysis. (This analysis is outlined in the Appendix as far as we have been able to take it.) Therefore, we have changed our methodology from direct to asymptotical analysis.

Strain Energy and Approximations

One of the most consistent ways to obtain the constitutive law for thin-body (*i.e.* beam and plate/shell) analysis is the use of asymptotical analysis. Literature for the successful analysis of asymptotical methods for beams can be found in Hodges (1987) and Atılgan and Hodges (1990) and for plates in Noor and Burton (1989). In addition to the *direct asymptotical* analysis, which is applied to *differential equations*, Berdichevsky (1978, 1979) developed the *variational-asymptotical* analysis, which is applied to *functionals*. Berdichevsky and his co-workers applied this method successfully to beams and shells for static and dynamic analysis. An outline and a simple application of this method to beam analysis can be found in Berdichevsky (1980). In what follows we will apply this method to non-homogeneous and anisotropic plates to obtain the most consistent approximations of the constitutive law.

Three-Dimensional Strain Energy for Anisotropic Plates

The three-dimensional strain energy for an anisotropic plate can be written as

$$U = \frac{1}{2} \int_A \int_h \Gamma^T D \Gamma dx_3 dA \quad (11)$$

where D is the 6×6 symmetric material stiffness matrix which relates the three-dimensional Biot strain Γ to the three-dimensional Jaumann stress Z . The form of this matrix can be found in Jones (1975) for all possible type of material structures ranging from transversely isotropic case to most general anisotropy. The Jaumann stress is also arranged in a 6×1 column matrix form $Z = [Z_{11} \ Z_{12} \ Z_{22} \ Z_{13} \ Z_{23} \ Z_{33}]^T$ so that

$$Z = D\Gamma \quad (12)$$

Since warping is a three-dimensional function of all coordinates, for most general configurations, it is not possible to deal with it only through the thickness. Therefore, one should discretize the warping as follows

$$w(x_1, x_2, x_3) = N(x_3) W(x_1, x_2) \quad (13)$$

Then, using this together with Eq. (8) in Eq. (11), one can obtain the strain energy as follows

$$U = \frac{1}{2} \int_A \left[\epsilon^T A \epsilon + \epsilon^T R^T W + W^T R \epsilon + W^T E W + \epsilon^T L_\alpha^T W_{,\alpha} + W_{,\alpha}^T L_\alpha \epsilon + W_{,\alpha}^T C_\alpha^T W + W^T C_\alpha W_{,\alpha} + W_{,\alpha}^T M_{\alpha\beta} W_{,\beta} \right] dA \quad (14)$$

where

$$\begin{aligned} A &= \int_h \mathcal{H}^T D \mathcal{H} dx_3 & R &= \int_h N'^T I_3^T D \mathcal{H} dx_3 & E &= \int_h N'^T I_3^T D I_3 N' dx_3 \\ L_\alpha &= \int_h N^T I_\alpha^T D \mathcal{H} dx_3 & C_\alpha &= \int_h N'^T I_3^T D I_\alpha N dx_3 & M_{\alpha\beta} &= \int_h N^T I_\alpha D I_\beta N dx_3 \end{aligned} \quad (15)$$

and where $()'$ denotes differentiation with respect to x_3 . Because the description of the displacement is 5 times redundant, the rigid-body portion of the warping degrees of freedom must be removed in forming these equations. After this, the matrix E will be positive definite. The rigid-body portion of w can be removed by constraining the finite element

nodes. In order to do this, we set $w_3 = 0$ at the reference surface and $w_\alpha = 0$ at the upper and lower surfaces.

Development of Finite Element Approximations for Anisotropic Plates

The *zeroth* approximation of the strain energy can be obtained by neglecting the warping completely. Then, the energy is only due to the global measures of the deformation and the stiffness matrix of the plate is just the matrix A . It can be shown that the matrix A gives an upper bound for the stiffnesses.

A first approximation of this functional can be obtained by taking only the first two lines of the strain energy functional. (This approximation cannot be called *the* first approximation until it is proven. We address this below.) These four terms are considered to be the most dominant terms in the strain energy functional. The reason for this is that differentiation with respect to the in-plane coordinates x_1 and x_2 will always result in smaller magnitudes than differentiation with respect to the thickness coordinate x_3 . Therefore, the energy obtained by the remainder terms should be smaller than the first four terms. The first approximation of the strain energy functional then reads

$$U^* = \frac{1}{2} \int_A (\epsilon^T A \epsilon + \epsilon^T R^T W + W^T R \epsilon + W^T E W) dA \quad (16)$$

Since at the beginning we consider warping to be an arbitrary quantity, independent of the global strain measures, the Euler-Lagrange equation associated with W will be obtained from this functional by taking a straightforward variations of with respect to W yielding

$$R\epsilon + EW = 0 \quad (17)$$

From this we obtain a relationship between our global strain measure ϵ and the warping W to be

$$W = -E^{-1} R\epsilon \quad (18)$$

In order to prove that this is *the* first approximation we need to find the second approximation. It can be shown by using a parallel development with that of Berdichevsky (1979) that the solution obtained here is the first approximation.

In order to find the stiffness matrix, Eq. (18) is used in the first approximation of the strain energy functional, Eq. (16) which gives

$$U^* = \frac{1}{2} \int_A \epsilon^T (A - R^T E^{-1} R) \epsilon dA \quad (19)$$

This is the the first approximation for the strain energy. Derivative of the strain energy with respect to the global strain measure ϵ results in the conjugate measures of the section stress resultants

$$Q = [F \quad M]^T = \left(\frac{\partial U^*}{\partial \epsilon} \right)^T \quad (20)$$

This relation suffices the existence of a relationship between stress resultants and the global measure of strains through an elastic law, which we call the first approximation of the *matrix of elastic stiffness constants*, S^* , as in Eq. (10)

$$Q = S^* \epsilon \quad (21)$$

Following the above operation, S^* can then be found as

$$S^* = A - R^T E^{-1} R \quad (22)$$

The matrix representing the first-order warping contribution to the stiffness matrix, $R^T E^{-1} R$, is positive definite. It can also be shown that a finite element approximation for the first asymptotic approximation S^* will be an upper bound on the actual stiffnesses.

The First Approximation for Isotropic Plates

When we reduce our equations to the isotropic case it is possible to obtain an analytical solution for warping and the stiffness matrix. Using Eq. (16) for the isotropic case (for the first approximation for isotropic case, it is not necessary to discretize the warping) gives the warping displacements as follows

$$\begin{aligned} w_1 &= w_2 = 0 \\ w_3 &= -2\nu \left[(\gamma_{11} + \gamma_{22}) x_3 + (\kappa_{11} + \kappa_{22}) \frac{x_3^2}{2} \right] \end{aligned} \quad (23)$$

where ν is Poisson's ratio. The the elastic stiffness constants can be expressed in terms of ν , Young's modulus E , and the shear modulus G . For the first approximation, the matrix of elastic constants is found to be

$$S^* = \begin{bmatrix} \frac{Eh}{(1-\nu^2)} & 0 & \frac{Eh}{(1-\nu^2)}\nu & 0 & 0 & 0 & 0 & 0 \\ 0 & Gh & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{Eh}{(1-\nu^2)}\nu & 0 & \frac{Eh}{(1-\nu^2)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Gh & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Gh & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{Eh^3}{12(1-\nu^2)} & 0 & \frac{Eh^3}{12(1-\nu^2)}\nu \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{Eh^3}{6(1+\nu)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{Eh^3}{12(1-\nu^2)}\nu & 0 & \frac{Eh^3}{12(1-\nu^2)} \end{bmatrix} \quad (24)$$

It can be seen that for the first approximation our results are exactly the same as those from classical isotropic plate theory. Since classical plate theory makes use of the *plane stress* assumption we see that the first approximation of our general functional also coincides with the plane stress assumption. If we were to neglect the warping starting from the beginning of our analysis then the stiffness values would be overestimated. The important point here is that *classical isotropic plate theory does include warping!*

As a first step in developing a numerical procedure, a finite element code has been written to evaluate the warping and the elastic constants for the isotropic case. Two-noded elements were used with C^0 continuous shape functions. Results which have been obtained coincide almost identically with classical theory, and the agreement becomes much better as more elements are taken. An example output from our finite element program is shown in Fig. 3. The distribution of warping through the thickness due to $\gamma_{11} + \gamma_{22}$ and $\kappa_{11} + \kappa_{22}$, respectively, are plotted in Figs. 4 and 5.

Elastodynamics of Beams

In this section, we describe a simplified case for longitudinal dynamics of a beam. We begin with the equations of motion and develop a weak form for mixed space-time finite elements. Finally, we present some numerical results.

Linear Rod Elastodynamic Equations

The equation of motion for a rod is given as

$$F' - \dot{P} + f = 0 \quad (25)$$

where F is the internal axial force, P is the linear momentum, and f is the force applied to the rod. All of these quantities are scalars and are functions of both space and time.

The linear rod kinematics is defined by a single displacement variable u . The velocity of the rod is obtained by differentiating the displacement with respect to time. Then the velocity of any point along the rod generator is

$$v = \dot{u} \quad (26)$$

The strain of the rod can be obtained by differentiating the displacement with respect to the spatial variable. So that the strain is

$$\epsilon = u' \quad (27)$$

The constitutive laws for the rod, which connect the strain and velocity to axial force and linear momentum, can be simply written as

$$F = \mu\epsilon \quad P = mv \quad (28)$$

The response of the rod can then be obtained by solving these equations simultaneously. Because of the simplicity of the structure, analytical solutions exist for some simple loadings. Thus, some initial and two-point boundary value problems can be used as benchmark cases in which space-time finite element method based upon above equations can be compared with some analytical solutions in the literature.

Weak Formulation for Space-Time Finite Elements

It is possible to obtain a weak formulation for a rod by just specializing the weak form given by Hodges (1990b). However, since for this simple model we have such a simple constitutive law, one may find it useful to satisfy the constitutive relationships (algebraic equations) strongly. In this way we do not introduce any more unknowns than are necessary. Consequently, the following weak form can be obtained

$$\begin{aligned} & \iint_{A(x,t)} \left(\delta \dot{u} - \frac{\delta P}{m} \right) P + \left(\frac{\delta F}{\mu} - \delta u' \right) F + u \left(-\delta \dot{P} + \delta F \right) + \\ & f \delta u + \oint_t (-P \delta u + u \delta P)|_x + \oint_x (F \delta u - u \delta F)|_t \end{aligned} \quad (29)$$

where δu , δP , and δF are test functions. It can be seen that one of the important properties of this functional is that none of the unknowns are ever differentiated. All the spatial and temporal differentiations are performed over the test functions. Weak forms with this property have been termed as "the weakest possible form" by Atılgan (1989). However, by allowing differentiation only over test functions yields a form such that the field equations now govern the test functions; note that they govern the trial functions in the primitive weak form. Since one may assign any function as a test function, the Green functions of the field equations could be chosen as test functions. The method using this kind of weak form has been called the "boundary element method" in the literature. Therefore, even though the weak forms can be same, selection of different shape functions can lead different solution strategies. This simple example can show that the differences in finite and

boundary element techniques are superficial; both are coming from the same background. More details along these lines and weak forms for theoretical mechanics will be found in a paper under preparation.

Applications

The initial value problem of a cantilevered rod subjected to a suddenly applied load (Heaviside step function) at the free end is considered. This is a classical wave propagation problem for which the force and linear momentum are discontinuous. This problem was also investigated by Iura *et al.* (1988) by using a different weak form and by Mansur and Brebbia (1984) by using the boundary element method.

Our space-time element is rectangular. With the weak form, Eq. (29), we have chosen the following shape functions. For u , F , and P , constants were chosen in the element interior. For the boundary, u is a constant but distinct value from the interior on each of the space and time boundaries. On the other hand, F and P are represented by Dirac delta functions at the element corners. The test functions δF and δP are linear in the space and time directions, respectively; and δu is bilinear in space and time directions. Our results are shown in Figs. 6 – 8 for the displacement, force, and linear momentum. Notice the discontinuous quantities are predicted accurately. Similar results for a case with initial displacement are shown in Figs. 9 – 11. In both cases, the results match the exact solution.

Future Work

In the near future we will develop the second approximation for plate analysis. When applied to the isotropic plate, this will result in a computational method for generating, as a check, the so-called “shear-correction factors.” After validation of the code, we will then extend it to treat anisotropic, laminated plate problems.

For the work on beam elastodynamics, we will continue to expand the capability of the analysis to deal with periodic excitation, arbitrary beam deformation, and nonlinear problems.

References

- Atilgan Ali R. (1989): Towards a Unified Analysis Methodology for Composite Rotor Blades. Ph. D. Dissertation, School of Aerospace Engineering, Georgia Institute of Technology, August.
- Atilgan, Ali R.; and Hodges, Dewey H. (1990): A Unified Nonlinear Analysis for Non-homogeneous, Anisotropic Beams with Closed Cross Sections. *AIAA Journal*, to appear.
- Atilgan, Ali R.; and Rehfield Lawrence W. (1990): Vibrations of Composite Thin-Walled Beams with Designed-in Elastic Couplings. *Composites '90: Recent Advances in Japan and the United States*, Ed. by Akira Kobayashi, will be published in October.
- Bajer, C. I.; and Bonthoux, C. G. (1988): State-of-the-art in True Space-Time Finite Element Method. *Shock and Vibration Digest*, vol. 20, no. 4, pp. 3 – 11.

Berdichevsky Victor, L. (1978): Variational-Asymptotic Method, (in Russian), *Some Problems of Continuum Mechanics*, Sedov's Anniversary Volume.

Berdichevsky Victor, L. (1979): Variational-Asymptotic Method of Constructing a Theory of Shells. *PMM*, vol. 43, no. 4, pp. 664 – 687.

Berdichevsky Victor, L. (1980): Variational-Asymptotic Method of Constructing the Nonlinear Shell Theory. *Theory of Shells*, Ed. by W. T. Koiter and G. K. Mikhailov, North-Holland Publishing Company, pp. 137 – 161.

Berdichevsky Victor, L. (1981): On the Energy of an Elastic Rod. *PMM*, vol. 45, no. 4, pp. 518 – 529.

Bert, Charles W. (1984): A Critical Evaluation of New Plate Theories to Laminated Composites. *Composite Structures*, vol. 2, pp. 329 – 347.

Borri, M., 1986: Helicopter Rotor Dynamics by Finite Element Time Approximation. *Computers and Mathematics with Applications*, Vol. 12A, No. 1, pp. 149 – 160.

Borri, Marco *et al.* (1985): Dynamic Response of Mechanical Systems by a Weak Hamiltonian Formulation. *Computers and Structures*, vol. 20, no. 1 – 3, pp. 495 – 508.

Borri, M., *et al.* (1988): Primal and Mixed Forms of Hamilton's Principle for Constrained and Flexible Dynamical Systems: Numerical Studies. ARO/AFOSR Workshop on Nonlinear Dynamics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia, June 1 – 3.

Danielson, Don A. (1989): Finite Rotation with Small Strain in Beams and Plates. Presented at the First Pan-American Congress on Applied Mechanics, Rio de Janeiro, Brazil, Jan.

Hodges, Dewey H. (1987): Nonlinear Beam Kinematics for Small Strains and Finite Rotations. *Vertica*, vol. 11, no. 3, pp. 573 – 589.

Hodges, Dewey H. (1990a): A Review of Composite Rotor Blade Modeling. *AIAA Journal*, vol. 28, no. 3, pp. 561 – 565.

Hodges, Dewey H. (1990b): A Mixed Variational Formulation Based on Exact Intrinsic Equations for Dynamics of Moving Beams. *International Journal of Solids and Structures*, vol. 26, no. 11, pp. 1253 – 1273.

Hodges, Dewey H.; Atilgan, Ali R., Fulton, Mark V.; and Rehfield, Lawrence W. (1989): Dynamic Characteristics of Composite Beam Structures. *Proceedings of the American Helicopter Society National Specialists' Meeting on Rotorcraft Dynamics*, Arlington, Texas, Nov. 13 – 14.

Iura, M; Borri, M.; and Atluri, S. N. (1988): Analysis of Travelling Wave Responses of Structures. *Computational Mechanics '88: Proceedings of the International Conference on Computational Engineering Science*, Ed. by Satya N. Atluri and G. Yagawa, vol. 2, pp. 44.v.1 – 44.v.5.

Giavotto, V. *et al.*,. (1983): Anisotropic Beam Theory and Applications. *Computers and Structures*, vol. 16, no. 1 – 4, pp. 403 – 413.

Librescu, Liviu; and Reddy, J. N. (1989): A Few Remarks Concerning Several Refined Theories of Anisotropic Composite Laminated Plates. *International Journal of Engineering Science*, vol. 27, no. 5, pp. 515 – 527.

Mansur, W. J.; and Brebbia, C. A. (1984): Solution of Wave Propagation Problems by Boundary Elements. *Boundary Element Methods in Engineering*, Ed. by C. A. Brebbia, vol. 1, pp. 170 – 186.

Noor, Ahmed K.; and Burton, Scott W. (1989): Assessment of Shear Deformation Theories for Multilayered Composite Plates. *Applied Mechanics Reviews*, vol. 41, no. 1, pp. 1 – 13.

Peters, David P.; and Izadpanah, Amir P. (1988): hp-version Finite Elements for the Space-time Domain. *Computational Mechanics*, vol. 3, no. 1, pp. 73 – 88.

Rehfield, Lawrence W.; and Atilgan, Ali R. (1989): On the Buckling Behavior of Thin Walled Laminated Composite Open Section Beams. AIAA Paper No. 89-1171, *Proceedings of the 30th Structures, Structural Dynamics and Materials Conference*, Mobile, Alabama, April 3 – 5, pp. 104 – 112.

Rehfield, Lawrence W.; and Atilgan, Ali R. (1989): Toward Understanding the Tailoring Mechanisms for Thin Walled Composite Tubular Beams. *First U.S.S.R. – U.S.A. Symposium on Mechanics of Composite Materials*, Ed. by S. W. Tsai, J. M. Whitney, T. W. Chou, and R. M. Jones, Published by ASME, pp. 187 – 196.

Rehfield, Lawrence W.; Atilgan, Ali R.; and Hodges, Dewey H. (1990): Nonclassical Behavior of Thin-Walled Composite Beams with Closed Cross Sections. *J. American Helicopter Society*, vol. 35, no. 2, Apr., pp. 42 – 50.

Reissner, Eric (1985): Reflections on the Theory of Elastic Plates. *Applied Mechanics Reviews*, vol. 38, no. 11, pp. 1453 – 1464.

Appendix

Consider a line element through the plate thickness (Fig. 12). The principle of virtual work for this filament can be written as follows

$$\left(\int_h \delta s^T Z_\alpha dx_3 \right)_{,\alpha} = \int_h \delta \Gamma^T Z dx_3 \quad (30)$$

where δs is the virtual displacement of an arbitrary point on the normal line element, $\delta \Gamma$ is a three-dimensional virtual strain, and Z is the three-dimensional stress measure conjugate to the strain, arranged in a 6×1 column matrix form $Z = [Z_{11} \ Z_{12} \ Z_{22} \ Z_{13} \ Z_{23} \ Z_{33}]^T$. The tractions on the lateral surfaces of the line elements are written as

$$Z_1 = [Z_{11} \ Z_{12} \ Z_{13}]^T \quad Z_2 = [Z_{22} \ Z_{12} \ Z_{23}]^T \quad (31)$$

This principle enforces the weak satisfaction of the three-dimensional equilibrium equations, and traction-free upper and lower surfaces. This is clearly analogous to the beam St. Venant problem, except that this is for a one-dimensional line element through the thickness of a plate.

Let us decompose the displacement field into warping and rigid body displacement components

$$s = w + u + x_3 \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} \quad (32)$$

where w and u are 3×1 column matrices for warping and the displacement of the reference surface, respectively. As outlined in the text, it is possible to find the three-dimensional strain as

$$\Gamma = \mathcal{H}\epsilon + I_3 w_{,3} + I_1 w_{,1} + I_2 w_{,2} \quad (33)$$

where matrices \mathcal{H} , I_3 , I_1 , and I_2 are defined in the text. The stress-strain relationship is given as $Z = D\Gamma$. Substitution of Eq. (33) into the principle of virtual work, and discretizing warping as $w = N(x_3)W(x_1, x_2)$ one can obtain the following system of equations

$$\begin{bmatrix} A & B & D & I \\ B^T & C & E & J \\ D^T & E^T & H & K \\ I^T & J^T & K^T & L \end{bmatrix} \begin{Bmatrix} W_{,1} \\ W_{,2} \\ W \\ \epsilon \end{Bmatrix} = \begin{Bmatrix} \mathcal{P}_1 \\ \mathcal{P}_2 \\ \mathcal{P}_{1,1} + \mathcal{P}_{2,2} \\ Q \end{Bmatrix} \quad (34)$$

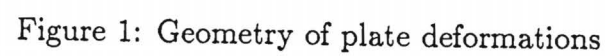
where

$$\begin{aligned} Q &= [F \quad M]^T & \mathcal{P}_\alpha &= \int_h N^T Z_\alpha dx_3 \\ F &= [F_{11} \quad F_{12} \quad F_{22} \quad F_{13} \quad F_{23}]^T & M &= [M_{11} \quad M_{12} \quad M_{22}]^T \end{aligned} \quad (35)$$

and the matrices are defined as

$$\begin{aligned} A &= \int_h N^T I_1^T D I_1 N dx_3 & B &= \int_h N^T I_1^T D I_2 N dx_3 \\ C &= \int_h N^T I_2^T D I_2 N dx_3 & D &= \int_h N^T I_1^T D I_3 N dx_3 & E &= \int_h N^T I_2^T D I_3 N dx_3 \\ H &= \int_h N^T I_3^T D I_3 N dx_3 & I &= \int_h N^T I_1^T D \mathcal{H} dx_3 \\ J &= \int_h N^T I_2^T D \mathcal{H} dx_3 & K &= \int_h N^T I_3^T D \mathcal{H} dx_3 & L &= \int_h \mathcal{H}^T D \mathcal{H} dx_3 \end{aligned} \quad (36)$$

Eq. (34) can be reduced to one matrix equation governing the warping. However, in order to solve that equation, one must find a scalar algebraic equation governed by each stress resultant (each element of Q). Since we have not been able to find such an equation, we have changed methods. It may be possible to find a solution for Eq. (34) by using a different approach, and this possibility is still open.



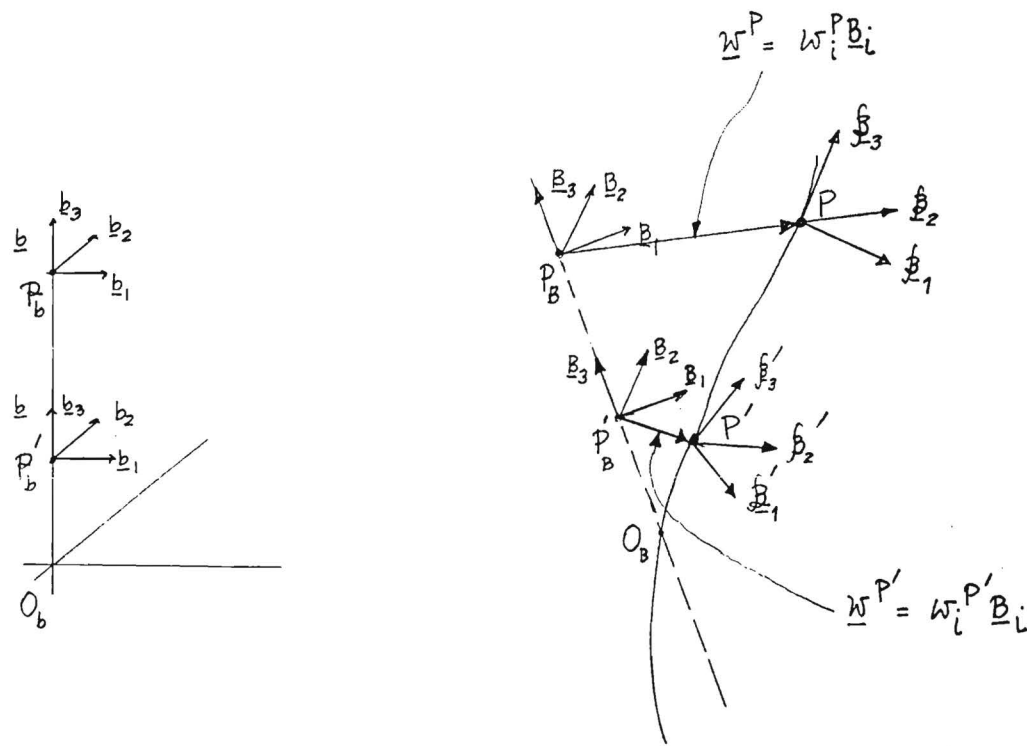


Figure 2: Concept of local rotation

Number of elements 32
 Thickness (m) 0.00200
 Young's modulus (N/m2) 0.2000D+09
 Poisson's ratio .25

Constraints : 1 2 51 97 98

A matrix (without warping)

0.4800D+06	0.0000D+00	0.1600D+06	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00
0.0000D+00	0.1600D+06	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00
0.1600D+06	0.0000D+00	0.4800D+06	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00
0.0000D+00	0.0000D+00	0.0000D+00	0.1600D+06	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00
0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.1600D+06	0.0000D+00	0.0000D+00	0.0000D+00
0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.1600D+00	0.0000D+00	0.5333D-01
0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.5333D-01	0.0000D+00
0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.5333D-01	0.0000D+00	0.1600D+00

$R^T E^{-1} R$ (warping effects)

0.5333D+05	0.0000D+00	0.5333D+05	0.0000D+00	0.0000D+00	0.1421D-13	0.0000D+00	0.1421D-13
0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00
0.5333D+05	0.0000D+00	0.5333D+05	0.0000D+00	0.0000D+00	0.1421D-13	0.0000D+00	0.1421D-13
0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00
0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00
0.1776D-13	0.0000D+00	0.1776D-13	0.0000D+00	0.0000D+00	0.1776D-01	0.0000D+00	0.1776D-01
0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00
0.1776D-13	0.0000D+00	0.1776D-13	0.0000D+00	0.0000D+00	0.1776D-01	0.0000D+00	0.1776D-01

S^* (stiffness matrix)

0.4267D+06	0.0000D+00	0.1067D+06	0.0000D+00	0.0000D+00	-0.1421D-13	0.0000D+00	-0.1421D-13
0.0000D+00	0.1600D+06	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00
0.1067D+06	0.0000D+00	0.4267D+06	0.0000D+00	0.0000D+00	-0.1421D-13	0.0000D+00	-0.1421D-13
0.0000D+00	0.0000D+00	0.0000D+00	0.1600D+06	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00
0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.1600D+06	0.0000D+00	0.0000D+00	0.0000D+00
-0.1421D-13	0.0000D+00	-0.1421D-13	0.0000D+00	0.0000D+00	0.1422D+00	0.0000D+00	0.3557D-01
0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.0000D+00	0.5333D-01	0.0000D+00
-0.1421D-13	0.0000D+00	-0.1421D-13	0.0000D+00	0.0000D+00	0.3557D-01	0.0000D+00	0.1422D+00

Figure 3: An example output from finite element code

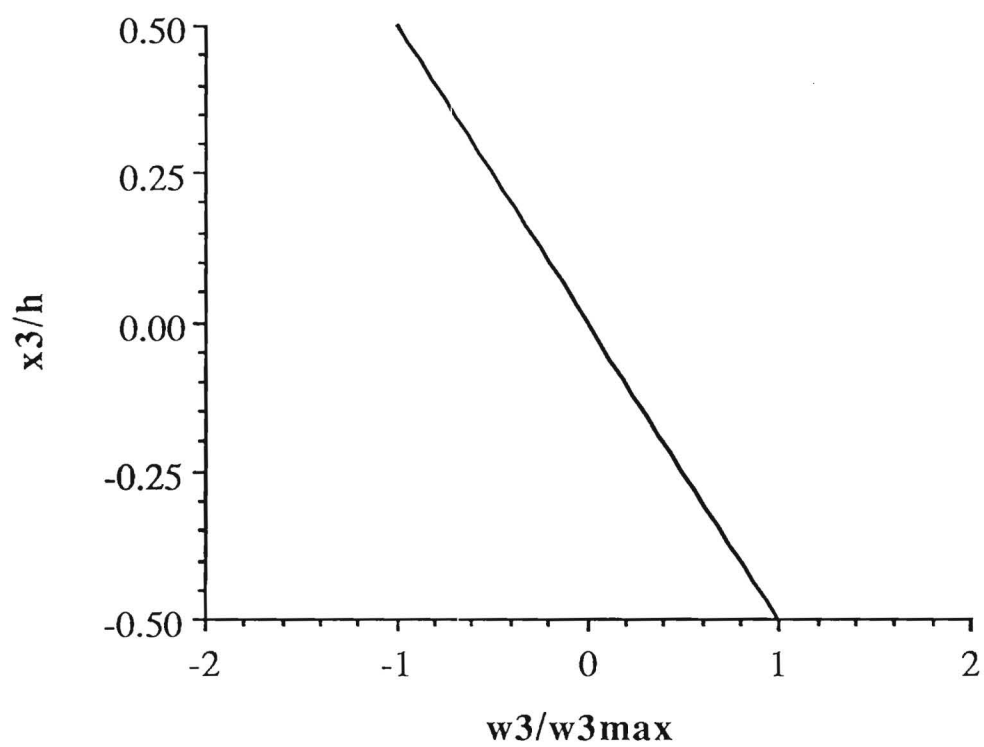


Figure 4: The distribution of warping through the thickness due to $\gamma_{11} + \gamma_{22}$

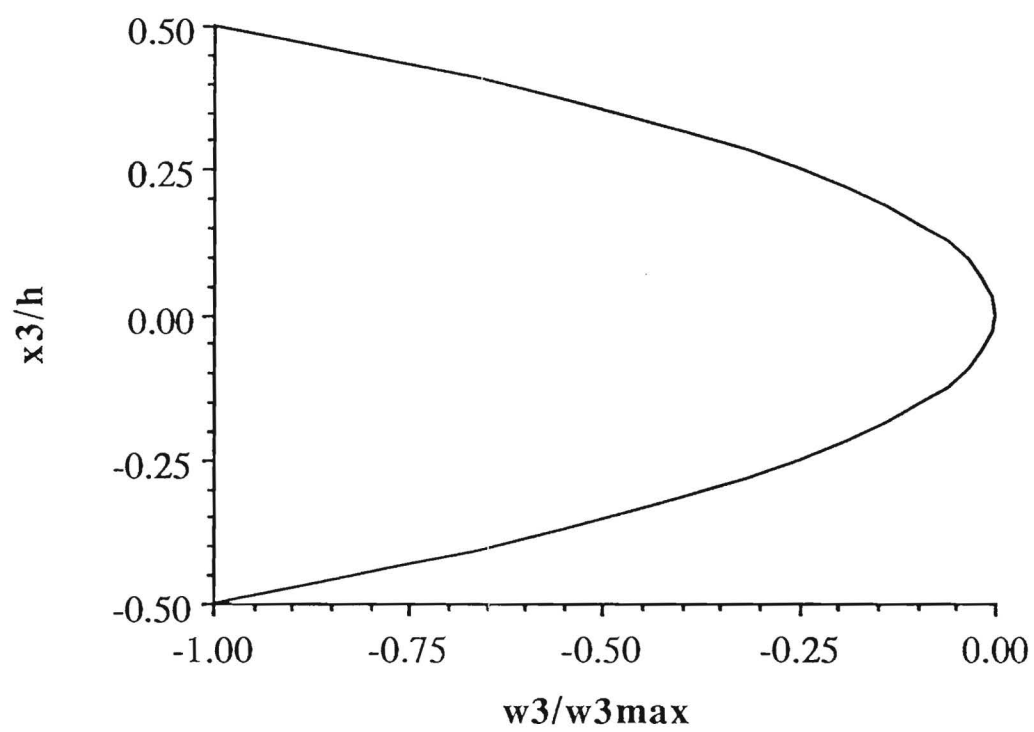


Figure 5: The distribution of warping through the thickness due to $\kappa_{11} + \kappa_{22}$

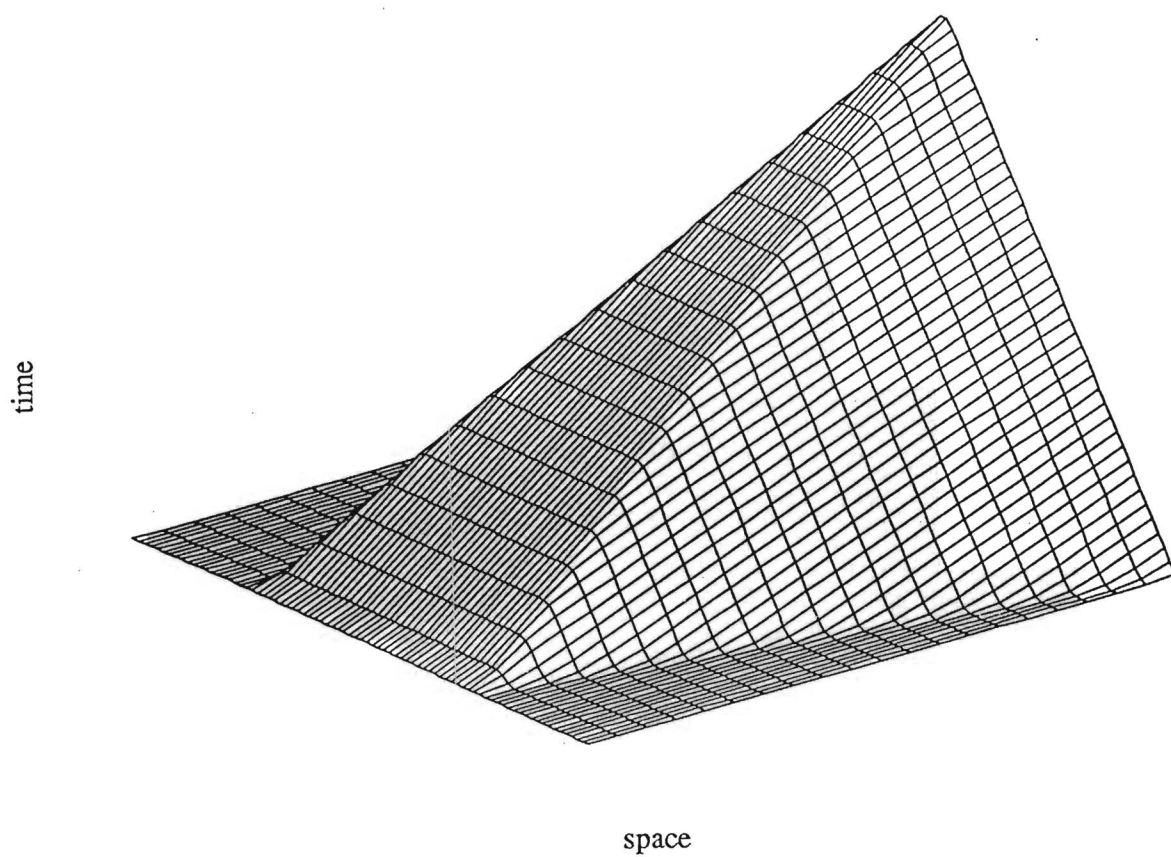


Figure 6: Displacement distribution in space-time domain due to heaviside step function

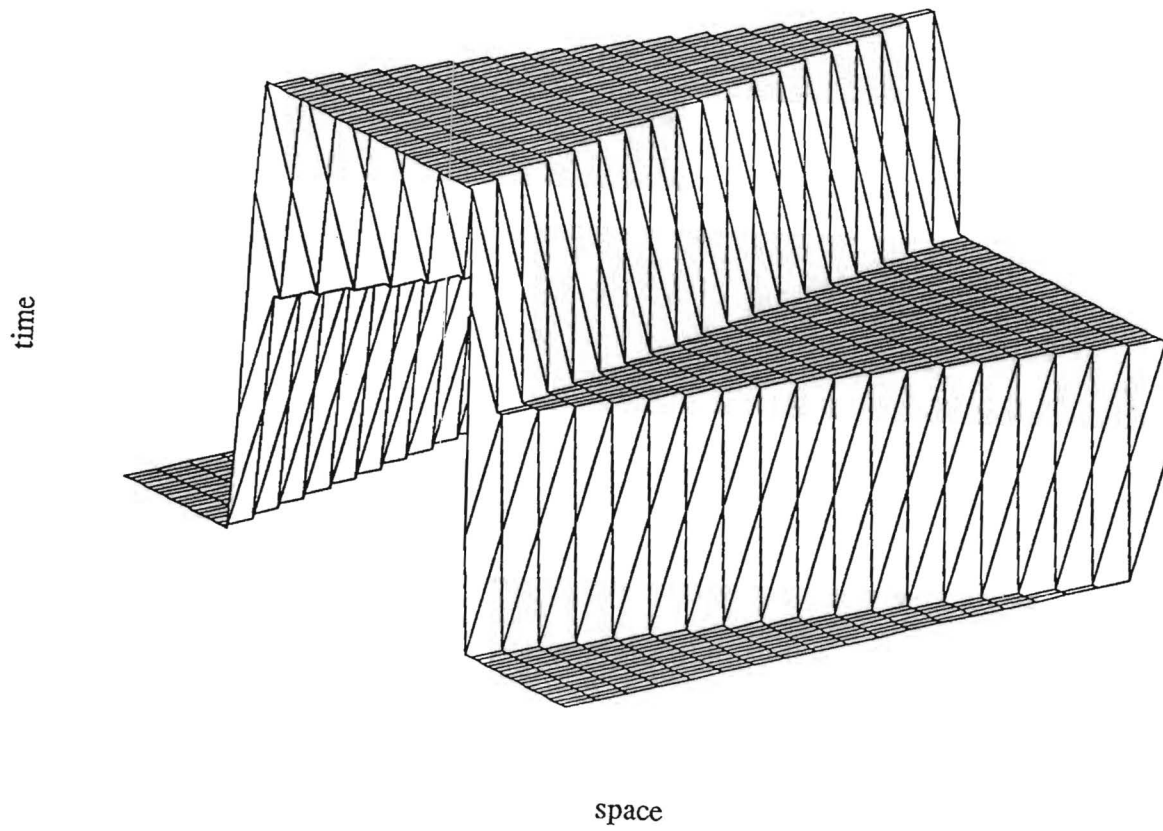


Figure 7: Force distribution in space-time domain due to heaviside step function

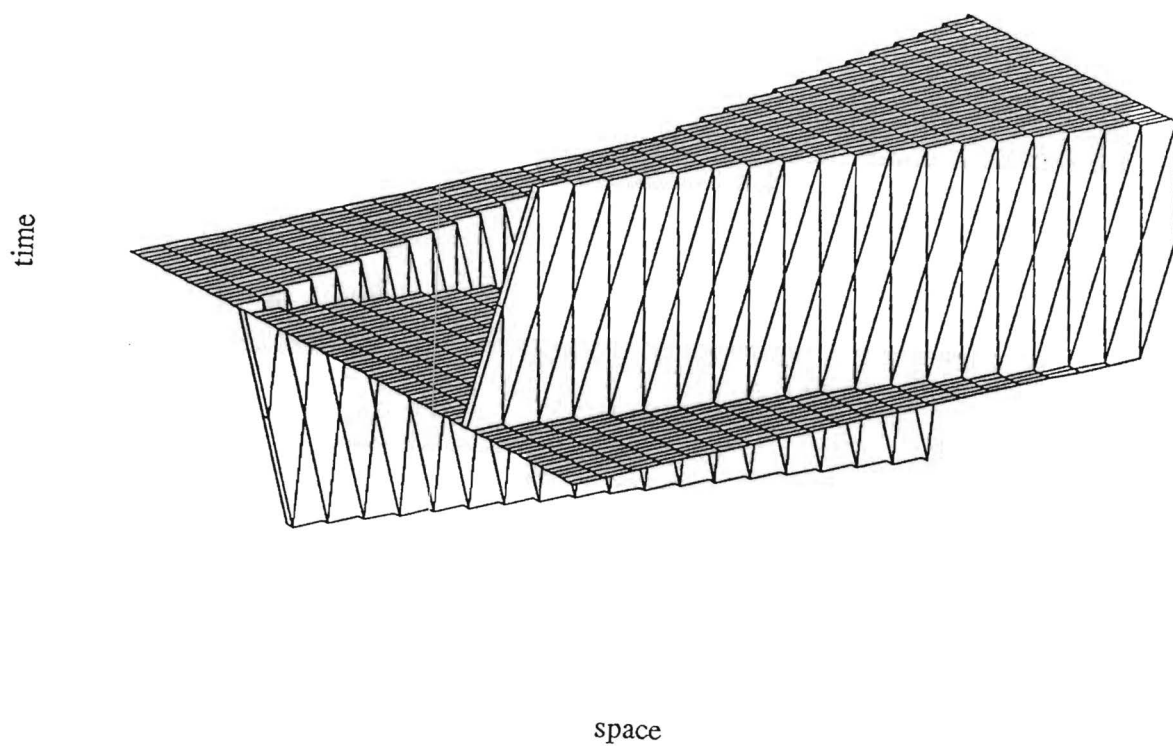


Figure 8: Linear momentum distribution in space-time domain due to heaviside step function

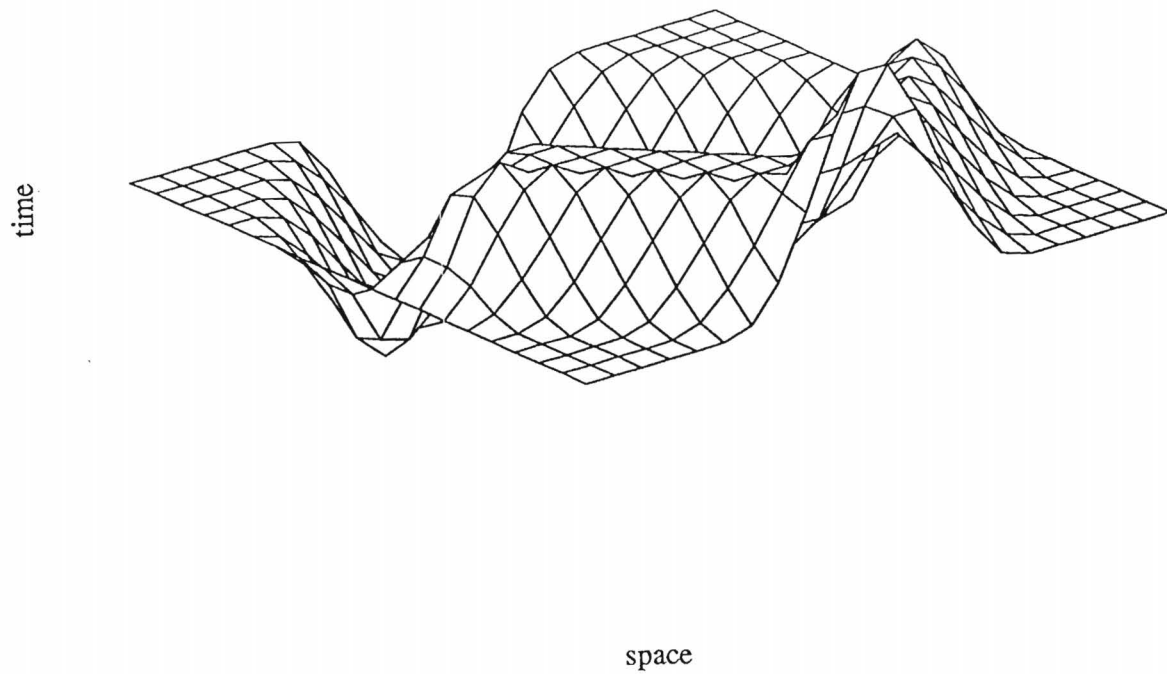


Figure 9: Displacement distribution in space-time domain due to initial displacement

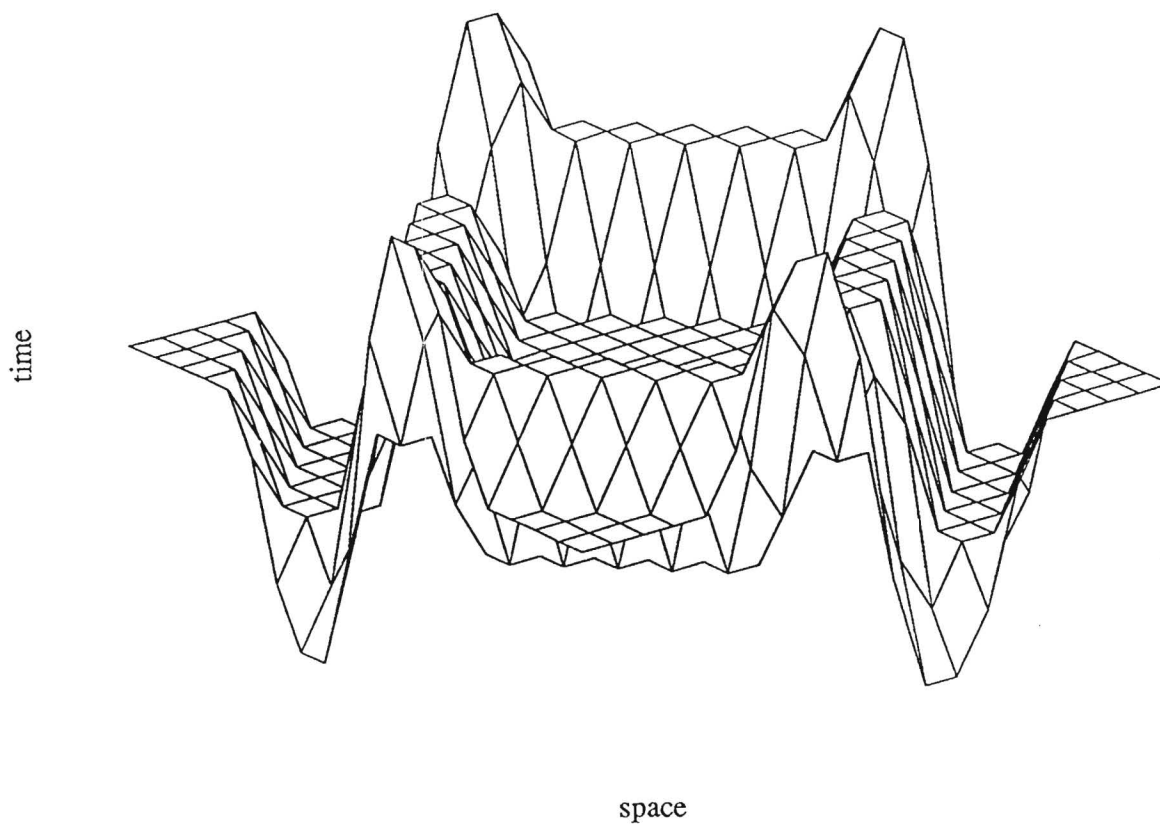


Figure 10: Force distribution in space-time domain due to initial displacement

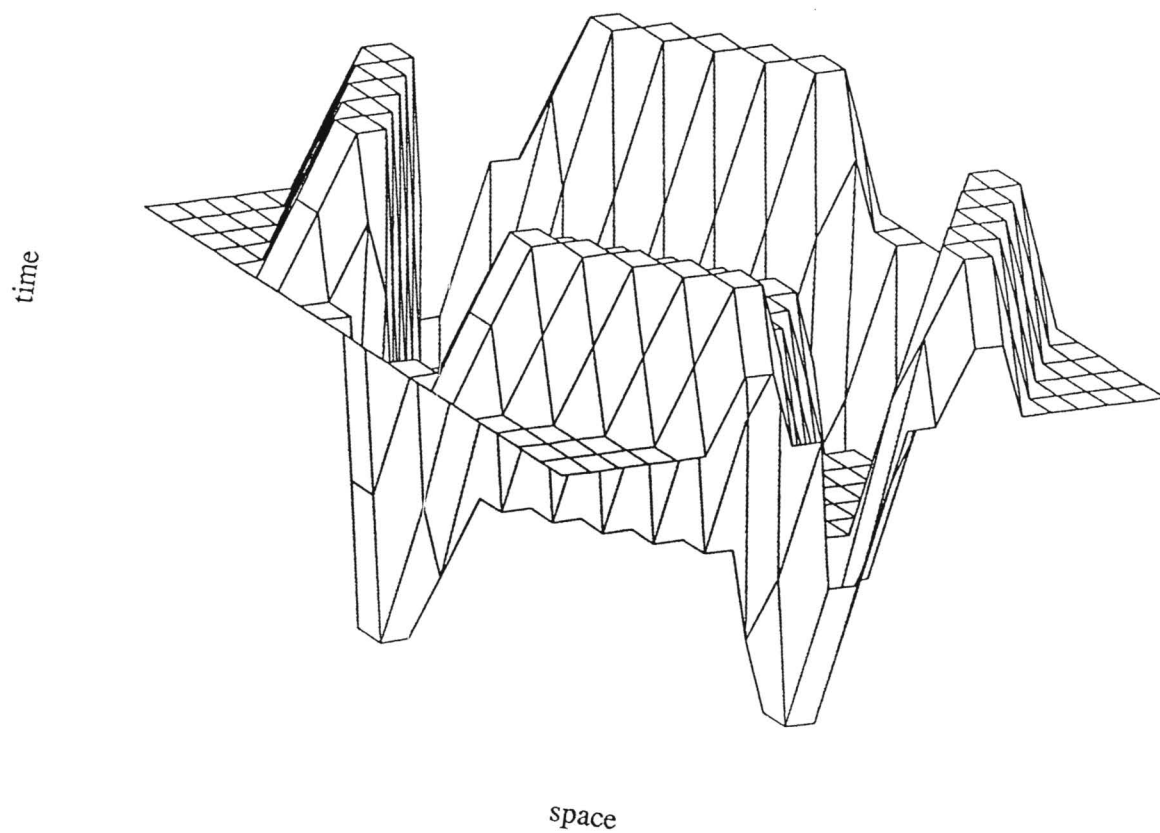
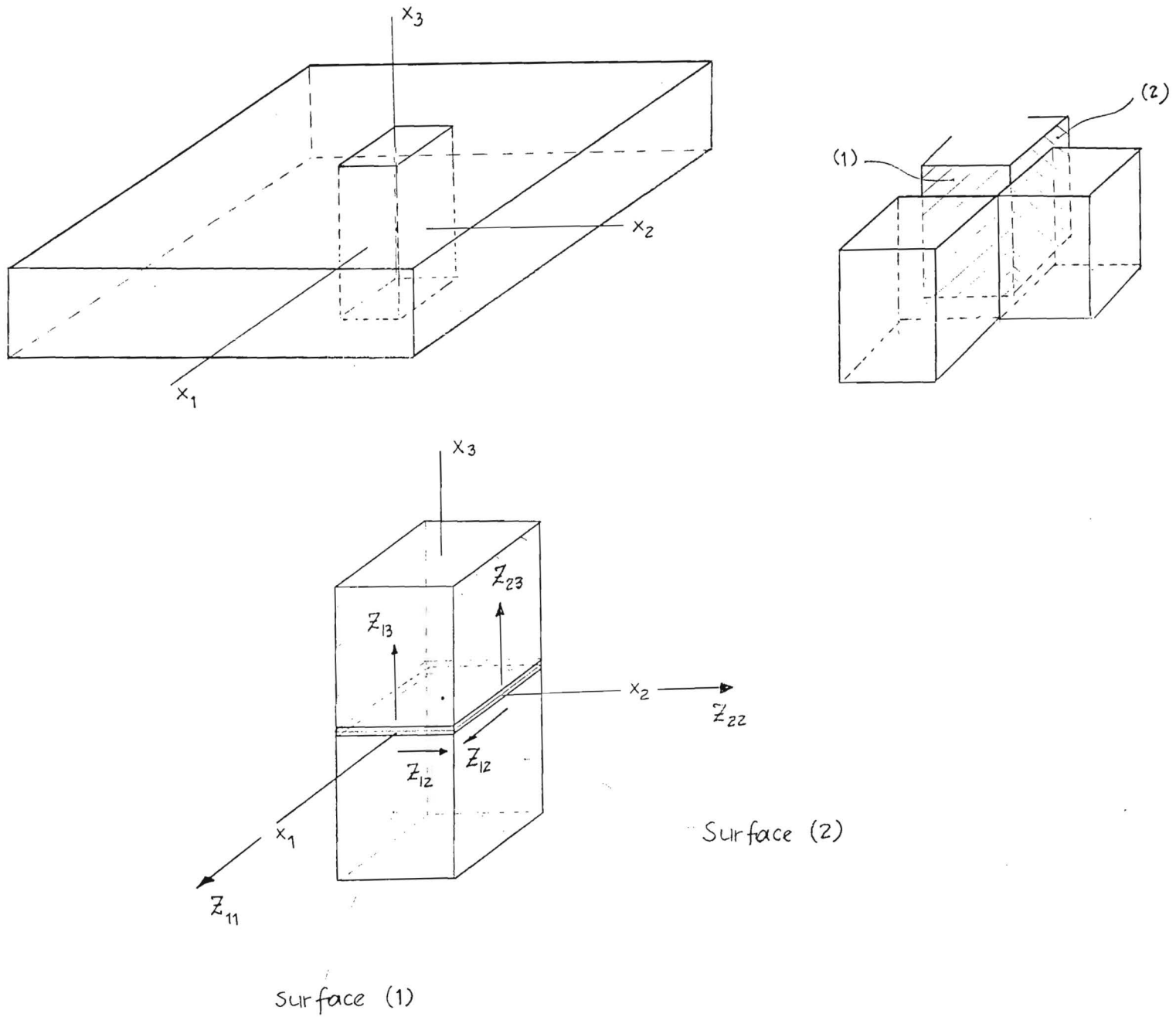


Figure 11: Linear momentum distribution in space-time domain due to initial displacement

Figure 12: An interior normal line element of a plate



Modeling of Composite Beams and Plates for Static and Dynamic Analysis

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Introduction

The purpose of this research is to develop a rigorous theory and corresponding computational algorithms for through-the-thickness analysis of composite plates. This type of analysis is needed in order to find the elastic stiffness constants for a plate and to post-process the resulting plate solution in order to find the warping, strain, and stress distributions through the thickness of the plate. In the last progress report, a theoretical approach and attendant finite element method were described, which we are continuing to follow. Additional insight has also been gained into the theory so that a finite element discretization may not be required in all cases. Both finite element and analytical methods have been pursued in parallel during this reporting period. Also during this period, the nonlinear global deformation analysis (deformation of the surface) of plates was addressed. This, together with the above, constitutes the complete boundary value problem for plates.

The present framework naturally leads to an exact development of equilibrium equations in terms of stress resultants and strain measures. Also, it is possible to express the strain measures exactly in terms of the derivatives of displacement and rotational variables. On the other hand, because of the nature of this two-dimensional formulation, "modeling" (reduction from three to two dimensions) is inherently approximate. In order to handle the modeling in the most consistent manner, asymptotical analyses are used.

Accomplishments to Date

In the first reporting period, we started with a "direct analysis" analogous to the beam modeling approach of Atılgan and Hodges (1991a). However, the resulting equations were not easy to solve, and this was abandoned (see the appendix of the first progress report). After that we turned our attention to the variational-asymptotical analysis of Berdichevsky (1980), which seemed to be a viable possibility for modeling. In fact, as we derived in the first report, the classical isotropic plate elastic constants are easily obtainable by this approach. We now summarize our progress in the second reporting period.

Modeling of a Plate

We investigated the modeling of laminated composite plates for the first approximation (in which shear deformation is ignored). Subject only to the restriction that the fiber directions in the undeformed plate are parallel to the plane of the plate, this approach leads to the same stiffnesses as would be obtained from classical laminated plate theory. (This restriction is subsequently referred to as the "lamination hypothesis.") Other characteristics of classical plate theory are also *consistently* obtained. The stress and strain distributions through the thickness are consistent with the plane stress hypothesis, and the concomitant stretching of the normal line element is automatically included – contrary to some "engineering" treatments in the literature in which the normal line remains rigid. Although we have begun to look at the second approximation (in which shear deformation and other effects are added), we will not consider the relaxation of the lamination hypothesis until much later in the project.

Next, we extended our modeling to the analysis of laminated plates in which transverse shear deformation effects are considered. This analysis corresponds to the second approximation in our variational-asymptotical analysis, and it is in agreement with the literature when the plate is either isotropic or anisotropic but homogeneous. In this case one can express the shear energy contribution in terms of a "shear-correction factor" given by $\frac{5}{6}$.

Finally, we considered plates that are both nonhomogeneous and anisotropic through the thickness. Even though the analysis turned out to be algebraically tedious, we have finally succeeded by virtue of the methodology outlined in Berdichevsky (1983). Therein, Berdichevsky performed his variational-asymptotical analysis for a special plate configuration where, although different materials are used, their Poisson ratios remain the same. We have extended his method to the case of a general laminated plate subject only to the lamination hypothesis. Our development is very simple, asymptotically correct, and *amenable to closed-form solutions*. This latter point means that finite element discretization may not be necessary. Our plan is to develop the finite element and the analytical methods in parallel in order to validate the results. We will then have the freedom to choose the method which is most convenient for rapid computation.

Rather than present theoretical details here, we have chosen to summarize the results here and include the details in the paper by Atilgan and Hodges (1991b). This paper will be finished in early February and sent under separate cover to the Technical Monitor. Results obtained from the finite element code agree with those from the analytical solution for a laminated composite plate. These results are given in Figs. 1 and 2 for a plate made of graphite-epoxy with a layup $(0, 90)_s$. The material properties are taken from Pagano (1969) and are given as

$$\begin{aligned} E_L &= 25 \times 10^6 \text{ psi} & E_T &= 10^6 \text{ psi} \\ G_{LT} &= 0.5 \times 10^6 \text{ psi} & G_{TT} &= 0.2 \times 10^6 \text{ psi} \\ \nu_{LT} &= \nu_{TT} = 0.25 \end{aligned}$$

where L signifies the direction parallel to the fibers and T the transverse direction. In the results below, the bonding between layers is assumed to be ideal. If interlaminar shear stress were to be calculated, however, the accuracy might be better if the layers of bonding material were modeled with separate elements.

The warping displacement for a laminated plate due to extension is shown in Fig. 1 and for pure bending in Fig. 2. Note that the warping is *not* analytic through the thickness. At each of the boundaries between laminae (at normalized thickness coordinates of ± 0.25 and zero), there exist discontinuities in the slope (Fig. 1) and curvature (Fig. 2) of the displacement function versus thickness. While these breaks appear to be rather minor, there are most certainly cases in which these breaks would be significant – especially in cases where the material matrix properties vary significantly through the thickness.

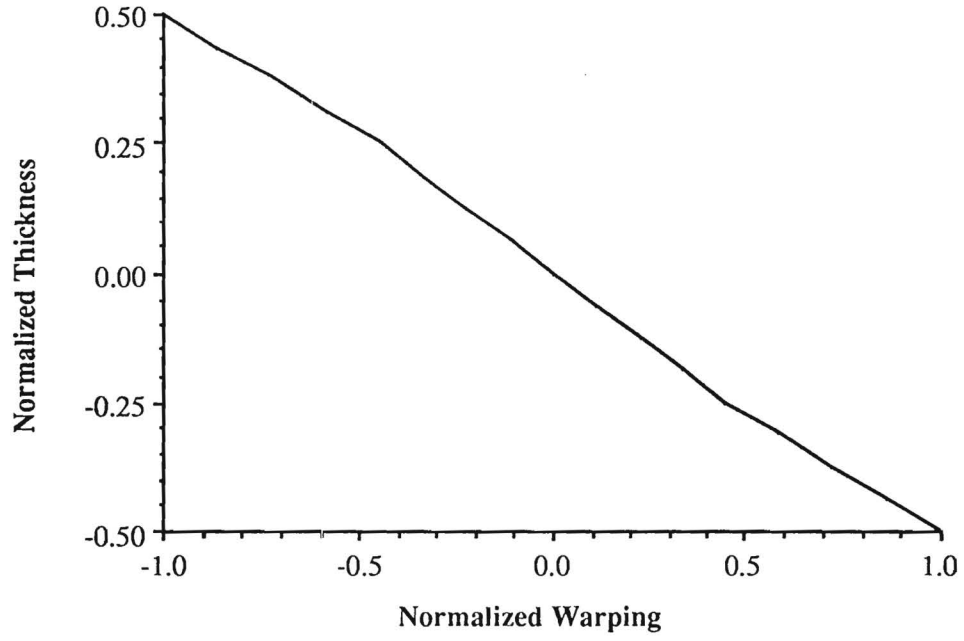


Fig. 1: Normalized warping distribution $w_3/w_{3\max}$ due to nonzero ϵ_{11} versus normalized thickness coordinate.

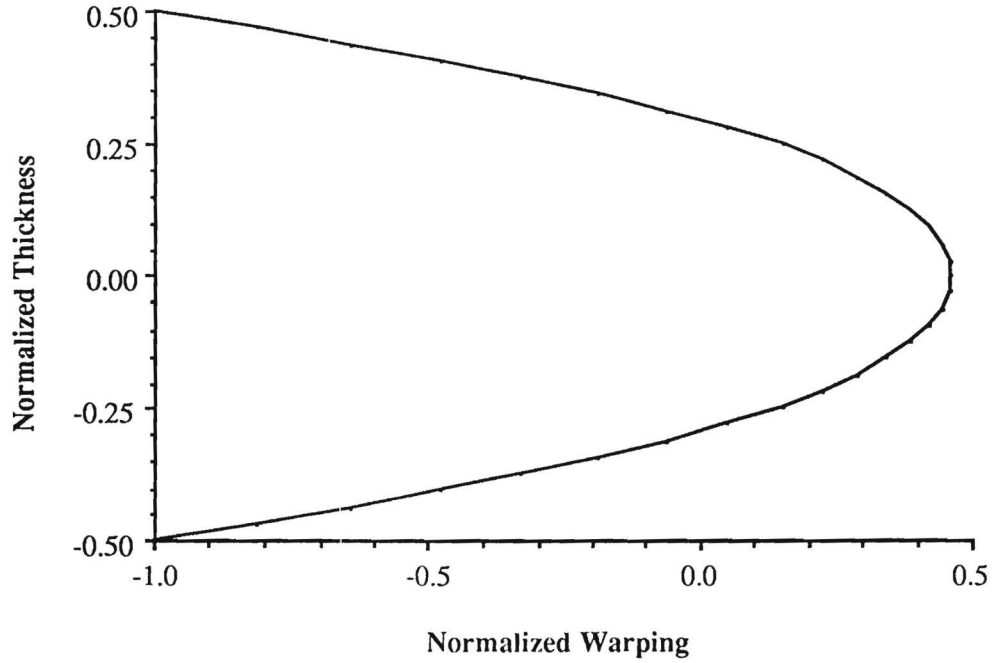


Fig. 2: Normalized warping distribution $w_3/w_{3\max}$ due to nonzero K_{11} versus normalized thickness coordinate.

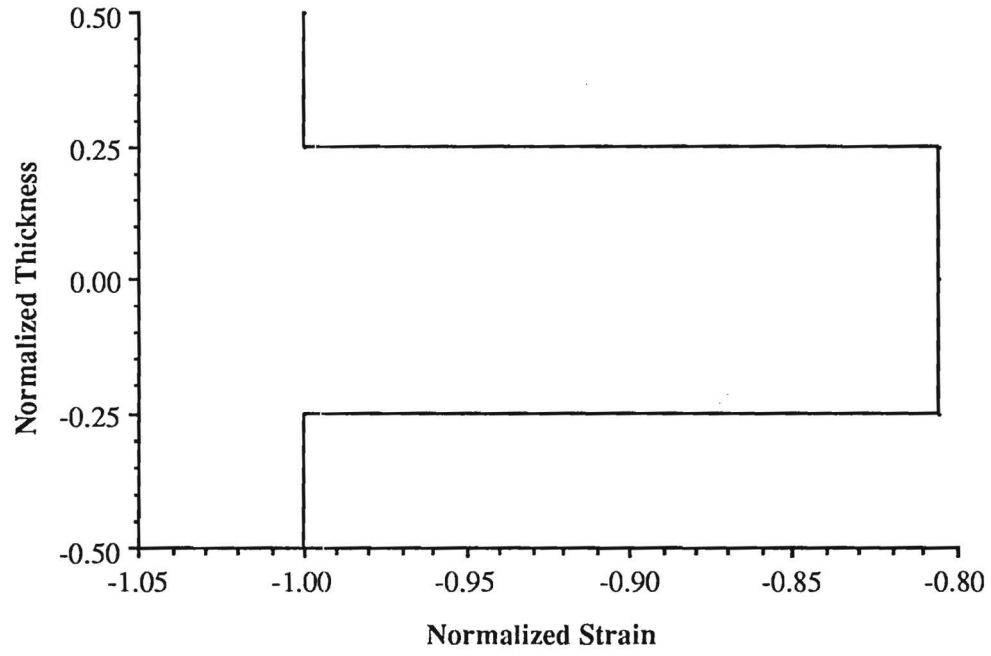


Fig. 3: Normalized strain distribution $\Gamma_{33}/\Gamma_{33_{\max}}$ due to nonzero ϵ_{11} versus normalized thickness coordinate.

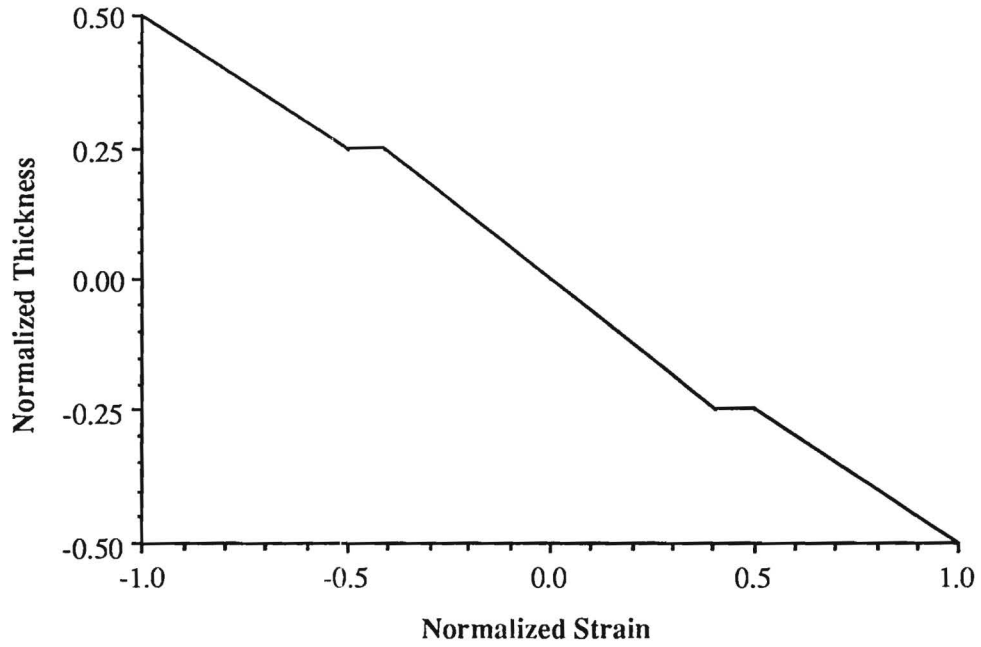


Fig. 4: Normalized strain distribution $\Gamma_{33}/\Gamma_{33_{\max}}$ due to nonzero K_{11} versus normalized thickness coordinate.

Also, as is well known, while these breaks in the derivatives of the displacement functions may appear minor, when these functions are differentiated to obtain the strain variation through the thickness, the breaks then become much more important. The strain variations through the thickness for uniform extension and bending cases are given in Figs. 3 and 4, respectively. The discontinuities present in both of these results are now very apparent and obviously important to take into account. The finite element results are exact with quadratic shape functions and one element per layer.

Global Deformation Analysis of a Plate

As far as global deformation analysis is concerned, the kinematics and intrinsic equilibrium equations were derived for plates undergoing large deflection and rotation but with small strain. The theory yields simple expressions for the strain field which agree with known results for special cases. It is shown that the rotation about the normal is not zero and can be expressed exactly in terms of other displacement and rotational variables. Exact intrinsic equilibrium equations, based on a form of the strain energy which follows from our modeling development, are derived using transpositional relations. The technical details of this work are contained in Hodges *et al.* (1990), which was sent to the Technical Monitor in December 1990.

Some Remarks on Edge-Zone Energy of a Plate

A likely criticism of the present work may be that we have labored to extend the analysis to include shear deformation while ignoring an equally if not more significant physical effect – the edge-zone phenomena. The strain energy associated with the edge-zone phenomena can be neglected in classical plate theory (the first approximation). On the other hand, taking into account the boundary energy is essential in the higher approximations in which the strain energy of a loaded edge depends on the self-equilibrated part of the load. Application of theories with corrections due to shear deformation without taking into account the boundary energy is somewhat inconsistent. Plates with free edges and with rigidly fixed boundaries, for which the boundary energy is found uniquely, are the exception.

We intend to look at the inclusion of edge-zone phenomena later in the project. During this reporting period we began to study the approaches in the literature to see which, if any, might be compatible with ours.

Elastodynamics of Beams

During this period we applied the mixed formulation as described in the last report to the forced response of transversely loaded beams. We are still studying the first results and are not yet prepared to present them publicly. Suffice it to say that we are encouraged by these results toward believing that space-time finite elements may be quite practical for such problems.

Future Work

In the next reporting period we intend to:

1. extend the finite element code to treat shear deformation;
2. extend the analytical solution to treat shear deformation;
3. validate the shear stiffness coefficients for a variety of problems with known solutions;
4. correlate with existing experimental results and exact (3-D) solutions for laminated plates with monoclinic material symmetry in each lamina and subjected to static loads;
5. study the error versus mesh type and density for various linear elastodynamics problems for beams.

In the long term we plan to:

1. develop modeling analysis for the most general nonhomogeneous, anisotropic plates where there may be fiber reinforcement through the thickness;
2. look into the boundary energy of a plate.

For work beyond the present project we are thinking about the development of fast, robust, and accurate computational techniques for the complete plate problem. This would include specifically the treatment of plate/beam connections.

References

- Atilgan, Ali R.; and Hodges, Dewey H. (1991a): A Unified Nonlinear Analysis for Nonhomogeneous, Anisotropic Beams with Closed Cross Sections. *AIAA Journal*, to appear.
- Atilgan, Ali R.; and Hodges, Dewey H. (1991b): On the Strain Energy of Laminated Composite Plates. To appear in the *Proceedings of the 32nd Structures, Structural Dynamics and Materials Conference*, Baltimore, Maryland, April 8 - 10.
- Berdichevsky Victor, L. (1980): Variational-Asymptotic Method of Constructing the Nonlinear Shell Theory. *Theory of Shells*, Ed. by W. T. Koiter and G. K. Mikhailov, North-Holland Publishing Company, pp. 137 - 161.
- Berdichevsky Victor, L. (1983): *Variational Principles of Continuum Mechanics*, (in Russian), Moscow, Nauka, pp. 301 - 313.
- Hodges, Dewey H.; Atilgan Ali R.; and Danielson, Donald A. (1990): On the Intrinsic Equations of Plate Theory. *Journal of Applied Mechanics*, submitted for publication.
- Pagano, N. J. (1969): Exact Solutions for Composite Laminates in Cylindrical Bending. *Journal of Composite Materials*, Vol. 3, No. 7, pp. 397 - 411.

Modeling of Composite Beams and Plates for Static and Dynamic Analysis

**Interim Semi-Annual Report
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Introduction

The purpose of this research is to develop a rigorous theory and corresponding computational algorithms for through-the-thickness analysis of composite plates.* This type of analysis is needed in order to find the elastic stiffness constants for a plate and to post-process the resulting plate solution in order to find the three-dimensional displacement, strain, and stress distributions throughout the plate.

In previous progress reports, a theoretical approach and attendant finite element method were described for the through-the-thickness analysis. Also, an analytical expression for the strain energy of nonhomogeneous, laminated composite plates was obtained by application of the variational-asymptotical method. Although these approaches were addressed in parallel, only the finite element had been coded. We have shown that the first approximation of this analysis method corresponds exactly to the classical laminated plate theory, and the second approximation to plate theories which account for shear deformation effects. No ad hoc shear correction factor is needed in our analysis. However, we did show that $\frac{5}{6}$ is the appropriate factor for homogeneous monoclinic plates in accordance with our theory. Finally, the geometrically nonlinear (possibly large deflection and rotation but with small strain) global deformation analysis of plates was addressed, and a paper was prepared based on that work. These ingredients comprise the complete boundary value problem for plates. The large deformation work is described in a paper by Hodges, Atilgan, and Danielson (1990), submitted last year for publication.

Summary of Work Done During This Period

Although we are continuing to develop and study both of the through-the-thickness methods, we have focused on evaluating the results of the variational-asymptotical method for the laminated plate during this reporting period. We also began to look into the reasons why the drilling moment is such a difficult problem in plate analysis. This is really outside the scope of the present research grant, and we plan to develop a separate research program in this area.

It should be noted that progress slowed considerably with the departure of Dr. Ali Atilgan to become Assistant Professor at Istanbul Technical University. Dr. Atilgan will continue to work with us on an informal basis. We have added one student to the project team who is getting up to speed with what is being done. We are currently seeking a post doctoral fellow to help complete the plate work.

Publication Activity

As described above, our approach consists of two separate domains, both of which are necessary in order to formulate the nonlinear three-dimensional problem. These are a

* Secondly, it originally concerned prediction of the dynamic behavior of beams using finite elements in space-time. This secondary aspect of the work is being carried out under the primary funding of NSF, and all results obtained from that work will be made available to the Aerostructures Directorate.

linear normal-line element analysis through the thickness and a nonlinear two-dimensional surface analysis. An exact development of intrinsic equilibrium equations was obtained in Hodges *et al.* (1990), and the three-dimensional strain energy in Atilgan and Hodges (1991a).

Large Deformation Analysis This reporting period we received rather critical reviews on the manuscript by Hodges *et al.* (1990). Thus, we continued to analyze the nonlinear global deformation analysis to attempt to deal with all the reviewers' criticisms. The main criticisms focused on how the moment equilibrium about the normal was described. In the revision, no equations needed to be changed; however, the wording has been changed to reflect the correct situation. Namely, moment equilibrium is implicitly enforced, but the moment equilibrium equation is not available from the energy. For details see the revised manuscript, which has now been resubmitted. A copy of the revised manuscript will be sent to the technical monitor before the end of July.

Variational-Asymptotical Analysis This work was described in a paper by Atilgan and Hodges (1991a), presented at the SDM this year and submitted for publication at that time. After submission, several typographical errors were found in the manuscript. More significantly, it was found that a term was inadvertently dropped which would contribute to the second approximation. If this term is kept, however, a closed-form approximate solution is not possible without introduction of additional variables, making the analysis more complicated. We believe that this term is negligible. Thus, validation becomes extremely important. If we are able to correlate well with the exact solution with the present theory it will mean that higher-degree expansions of the displacement or large numbers of extra variables in the layer-wise theories are really not necessary. We have recently received reviews for this paper. These, however, were rather benign by comparison with the above. The paper has now been revised and resubmitted. This revised manuscript, Atilgan and Hodges (1991b), will also be sent to the technical monitor before the end of July.

Validation

The initial form of validation we have chosen for the method is to generate cylindrical bending solutions that can be compared with the exact solution of Pagano (1969), similar to the work of Tessler and Saether (1990) and Cho (1991). In this report, the displacement and stress distributions through the thickness of a homogeneous orthotropic plate will be compared to the known exact elasticity solution. We are in the process of doing the same thing for the laminated case, but there are still errors in the code that preclude us from presenting any results for that case. Herein we will describe only the necessary terms for cylindrical bending from the general formulations reported previously. Notation follows that of our two papers described above.

Global Analysis Let's consider an infinitely wide plate of thickness h and length L where the x_1, x_2, x_3 axes are aligned to the direction of length, infinite width, and thickness;

respectively. Kinematical equations can be reduced from Hodges *et al.* (1990) for the linear case as

$$\begin{aligned}
\epsilon_{11} &= u_{1,1} \\
2\epsilon_{12} &= u_{1,2} + u_{2,1} \\
\epsilon_{22} &= u_{2,2} \\
\kappa_{11} &= \alpha_{1,1} \\
2\kappa_{12} &= \alpha_{1,2} + \alpha_{2,1} \\
\kappa_{22} &= \alpha_{2,2} \\
2\gamma_{13} &= \alpha_1 + u_{3,1} \\
2\gamma_{23} &= \alpha_2 + u_{3,2}
\end{aligned} \tag{1}$$

Similarly, equilibrium equations are

$$\begin{aligned}
N_{11,1} + N_{12,2} + f_1 &= 0 \\
N_{12,1} + N_{22,2} + f_2 &= 0 \\
Q_{1,1} + Q_{2,2} + f_3 &= 0 \\
M_{11,1} + M_{12,2} - Q_1 &= 0 \\
M_{12,1} + M_{22,2} - Q_2 &= 0
\end{aligned} \tag{2}$$

Finally, the constitutive law from Atilgan *et al.* (1991) can be expressed as

$$\begin{Bmatrix} N \\ M \\ Q \end{Bmatrix} = \begin{bmatrix} A & B & 0 \\ B^T & D & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \epsilon \\ K \\ \gamma \end{Bmatrix} \tag{3}$$

The plate is subject to sinusoidal surface loading of the form

$$f_3(x_1) = p_0 \sin\left(\frac{\pi x_1}{L}\right) \tag{4}$$

and $f_1 = f_2 = 0$. The loading f_3 is assumed to be imposed in the form of an upper surface traction.

Since the plate is infinite in the x_2 direction, all derivatives with respect to x_2 are zero. Plate boundary conditions for the simply supported case are

$$u_1(0) = N_{11}(L) = u_3(0) = u_3(L) = M_{11}(0) = M_{11}(L) = 0 \tag{5}$$

The triad \mathbf{B}_i can be derived by adopting the plate geometry described in Hodges *et al.* (1990) in terms of displacements and rotations with respect to the reference triad \mathbf{b}_i for cylindrical bending case.

$$\begin{aligned}
\mathbf{B}_1 &= \mathbf{b}_1 + \frac{u_{2,1}}{2} \mathbf{b}_2 - \alpha_1 \mathbf{b}_3 \\
\mathbf{B}_2 &= -\frac{u_{2,1}}{2} \mathbf{b}_1 + \mathbf{b}_2 \\
\mathbf{B}_3 &= \alpha_1 \mathbf{b}_1 + \mathbf{b}_3
\end{aligned} \tag{6}$$

Through-the-thickness Analysis The material properties are taken from Pagano (1969) and are given as

$$\begin{aligned}
E_\ell &= 25 \times 10^6 \text{ psi} & E_T &= 10^6 \text{ psi} \\
G_{\ell T} &= 0.5 \times 10^6 \text{ psi} & G_{TT} &= 0.2 \times 10^6 \text{ psi} \\
\nu_{\ell T} &= \nu_{TT} = 0.25
\end{aligned}$$

where ℓ signifies the direction parallel to the fibers and T the transverse direction.

After some calculation based on Atilgan and Hodges (1991b), the displacement field

$$\mathbf{z} = z_i \mathbf{b}_i = \hat{\mathbf{R}} - \hat{\mathbf{r}} \tag{7}$$

can be obtained for a homogeneous monoclinic plate in terms of the global strain measures, ϵ , 2γ , and K , as

$$\begin{aligned}
z_1 &= u_1 + h\zeta\alpha_1 + \frac{h^2}{2} \left(\zeta^2 - \frac{1}{12} \right) D_{\perp}^1 \epsilon_{11,1} + \left(\frac{h^3}{6} D_{\perp}^1 K_{11,1} - \frac{5h}{3} 2\gamma_{13} \right) \left(\zeta^3 - \frac{3}{20} \zeta \right) \\
z_2 &= u_2 - \frac{5}{3} h \left(\zeta^3 - \frac{3}{20} \zeta \right) 2\gamma_{23} \\
z_3 &= u_3 + 12h \left(E_{\perp 1}^1 \epsilon_{11,1} + h E_{\perp 2}^1 K_{11,1} \right) - h\zeta D_{\perp}^1 \epsilon_{11} - \frac{h^2}{2} \left(\zeta^2 - \frac{1}{12} \right) D_{\perp}^1 K_{11}
\end{aligned} \tag{8}$$

where the superscript of D_{\perp}^i , $E_{\perp 1}^i$, and $E_{\perp 2}^i$ represents the i th element of the matrices. Here the elements of ϵ , K , and 2γ can be found from the global deformation analysis. Similar expressions for all the strain and stress components can be obtained.

We evaluated the displacement and corresponding stress and strain distributions for a homogeneous orthotropic plate for our second approximation and compared with the exact solution of Pagano (1969). The ratio of length to thickness is taken as 4. Since this represents a relatively thick plate, the shear stresses play an important role in the behavior of the plate. (Note that the solutions converge to the results of the first approximation as this value increases. Namely, as the plate becomes thinner, the assumption of the normal element remaining straight becomes more reasonable.) For plotting the displacement and stress quantities, the following normalized parameters are used

$$\bar{Z}_{ij} = \frac{Z_{ij}(x_1, \zeta)}{p_0} \quad \bar{z}_i = \frac{E_T z_i(x_1, \zeta)}{h p_0}$$

In Figs. 1 - 4, we show the exact solution compared with the first approximation (classical theory) and our second approximation. The distribution of the longitudinal displacement \bar{z}_1 through the thickness at $x_1 = 0$ is shown in Fig. 1. Notice that this curve is very much like results for the corresponding beam problem. Fig. 2 shows the stress distribution \bar{Z}_{11} through the thickness at $x_1 = \frac{L}{2}$. This exhibits the continuous nonlinear distribution. As mentioned above, if $\frac{h}{L}$ decreases, this result converges to the first approximation. In both cases, the second approximation agrees very well with the exact solution. Fig. 3 shows the transverse shear stress distribution \bar{Z}_{13} through the thickness at $x_1 = 0$. Here the transverse shear stress is obtained from both using the constitutive equation (CE) and from integration of the equation of equilibrium of elasticity (EE). Also shown is the solution of the first approximation of current theory by using equilibrium equation. Notice that the first approximation of \bar{Z}_{13} using equilibrium equation is essentially identical with the second approximation using the constitutive relationship. Also, the second approximation using the equilibrium equations is very close to the exact solution. The exact solution distributions are not exactly symmetric with respect to mid-surface. For interpretation of this characteristic, the distribution of the transverse normal stress at $x_1 = \frac{L}{2}$ is plotted in Fig 4. Since no traction exists at the bottom of the plate, while loading is applied at the top surface of the plate, the asymmetric behavior can be rationalized.

We have begun to validate the theory for laminated plates, but there are still some errors in our computer code. This work should be completed during the next reporting period.

Drilling Degrees of Freedom

Concerning the drilling moment, it is absolutely essential to recognize that the difficulty of including nodes with actual drilling degrees of freedom in a plate formulation is not a mathematical one. That is, one cannot "fix" this problem totally by coming up with a clever choice of shape functions. Likewise, the answer does not lie in going to a mixed (versus displacement) formulation. *A point drilling moment applied to a plate produces an infinite drilling rotation, because the physics embodied in the theory of elasticity do not resist couple stresses.* Indeed, point moments in a three-dimensional elastic medium also result in infinite rotations.

Nodal moments in a beam theory are resisted in all directions, because the one-dimensional model has couple stresses implicit in it. These come into the analysis by virtue of integration of the stresses over the cross section of the beam. On the other hand, plates have couple stresses only through the thickness. Thus, a point moment parallel to the plane of the plate will be resisted. A point moment normal to the plate, however, is not in a direction that the medium can resist because there are no couple stresses associated with that direction. A sandwich plate theory which does resist drilling moments can be

found in Reissner (1972), who introduced a normal line element which resists twisting moments (about the normal). With this type of theory, a point drilling moment would result in a finite rotation about the normal, because the physics are appropriate.

If one stays in the context of the theory of elasticity (without couple stresses in the three-dimensional formulation), then a way to deal with connections between beams and plates and between plates at oblique angles to each other is needed. These connections are not trivial. Such structures intersect each other over a two-dimensional surface. Take a beam of circular cross section perpendicular to a circular plate – having a common axis of symmetry. If the beam has a nonzero radius and is subjected to an external twisting moment, it, in turn, imparts a radially varying traction over a circular area to, say, the upper surface of the plate. The average rotation of this area of the plate is finite. In the limit as the radius of the beam tends toward zero, however, the rotation does become infinite. So, the plate must “know” something about the size of the beam in order to know how to respond. The beam, on the other hand, really doesn’t care about the plate’s properties. The same type of conclusion can be drawn if we consider a circular plate with a circular hole at the center. Loading the inside of the hole with tractions so as to put a drilling moment on the plate centered at the hole, one finds that, as long as the hole has a radius greater than zero, the average rotation is finite.

The upshot of all this is that these connections are tough problems. It may be possible within the context of the asymptotical methods we have been working with to give an accurate but efficient approximation to these problems. Another student working on this would help, but for this another grant would be needed.

Future work

In the near future we intend to:

1. validate the analytical solution for symmetric, anti-symmetric, and general (asymmetric) laminated plates;
2. begin work on a modeling analysis for the most general type of nonhomogeneous, anisotropic plate
3. begin work on a finite element analysis for global deformation of plates.

In the long term we plan to:

1. look into the boundary energy of plates;
2. work toward fast, robust, and accurate computational techniques for the complete plate problem, including specifically the treatment of plate/beam connections and plate/plate connections at oblique angles. (beyond the scope of the present project)

References

Atilgan, Ali R.; and Hodges, Dewey H. (1991b): On the Strain Energy of Laminated Composite Plates. *Proceedings of the 32nd Structures, Structural Dynamics and Materials Conference*, Baltimore, Maryland, April 8 – 10, pp. 996 – 1007 (AIAA Paper 91-0960).

- Atilgan, Ali R.; and Hodges, Dewey H. (1991): On the Strain Energy of Laminated Composite Plates. *International Journal of Solids and Structures*, submitted for publication.
- Cho, M. (1991): An Efficient Higher-Order Composite Plate Theory for General Lamination Configurations. *Proceedings of the 32nd Structures, Structural Dynamics and Materials Conference*, Baltimore, Maryland, April 8 – 10, pp. 1496 – 1510 (AIAA Paper 91-1204).
- Hodges, Dewey H.; Atilgan, Ali R.; and Danielson, Donald A. (1990): A Geometrically Nonlinear Theory of Elastic Plates. *Journal of Applied Mechanics*, submitted for publication.
- Pagano, N. J. (1969): Exact Solutions for Composite Laminates in Cylindrical Bending. *Journal of Composite Materials*, Vol. 3, No. 7, pp. 397 – 411.
- Reissner, E. (1972): On Sandwich-Type Plates with Cores Capable of Supporting Moment Stresses. *Acta Mechanica*, Vol. 14, pp. 43 – 51.
- Tessler, A.; and Saether, E. (1990): A Computationally Viable Higher-order Theory for Laminated Composite Plates. *International Journal for Numerical Methods in Engineering*, Vol. 31, pp. 1069 – 1086.

Longitudinal Displacement at End

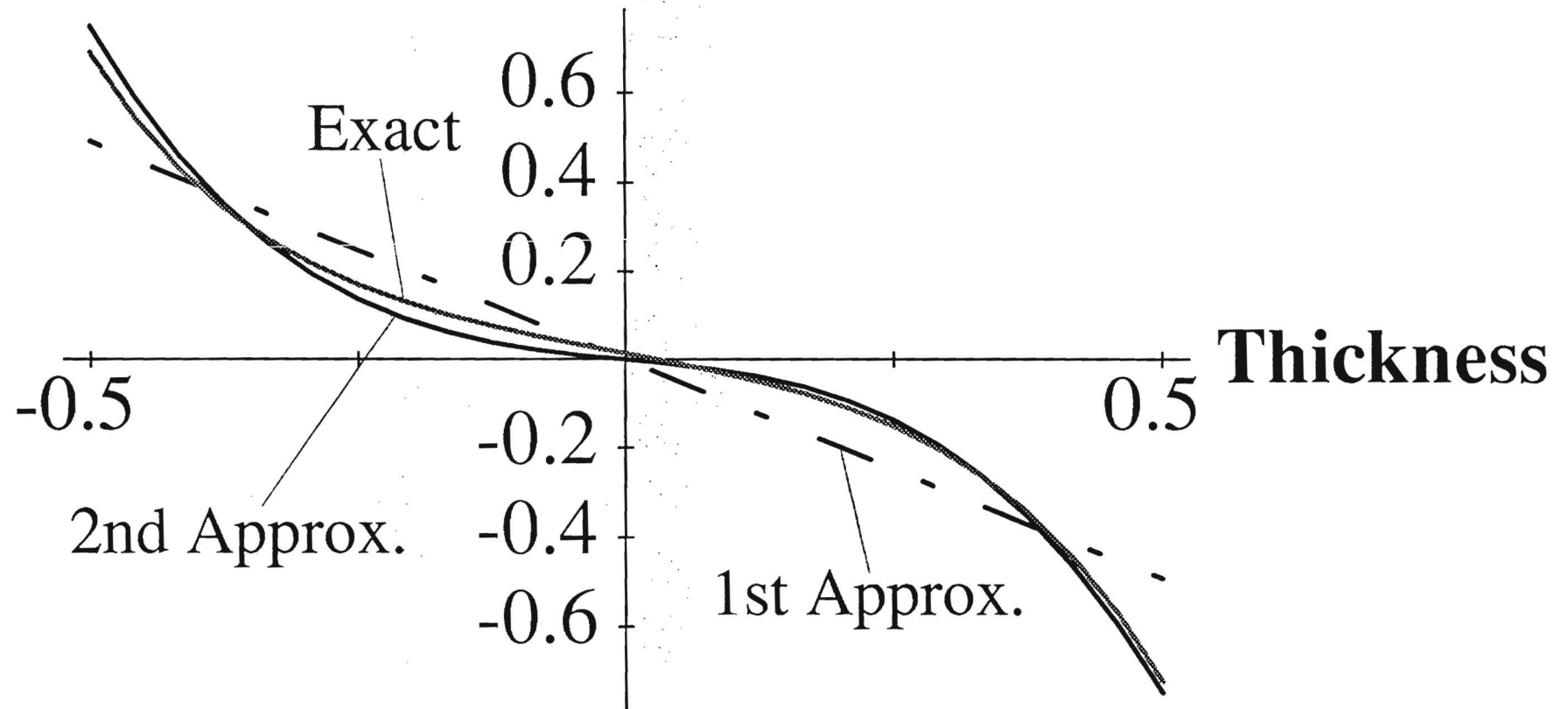


Fig. 1: Normalized in-plane displacement distribution \bar{z}_1 versus normalized thickness coordinate.

Longitudinal Stress at L/2

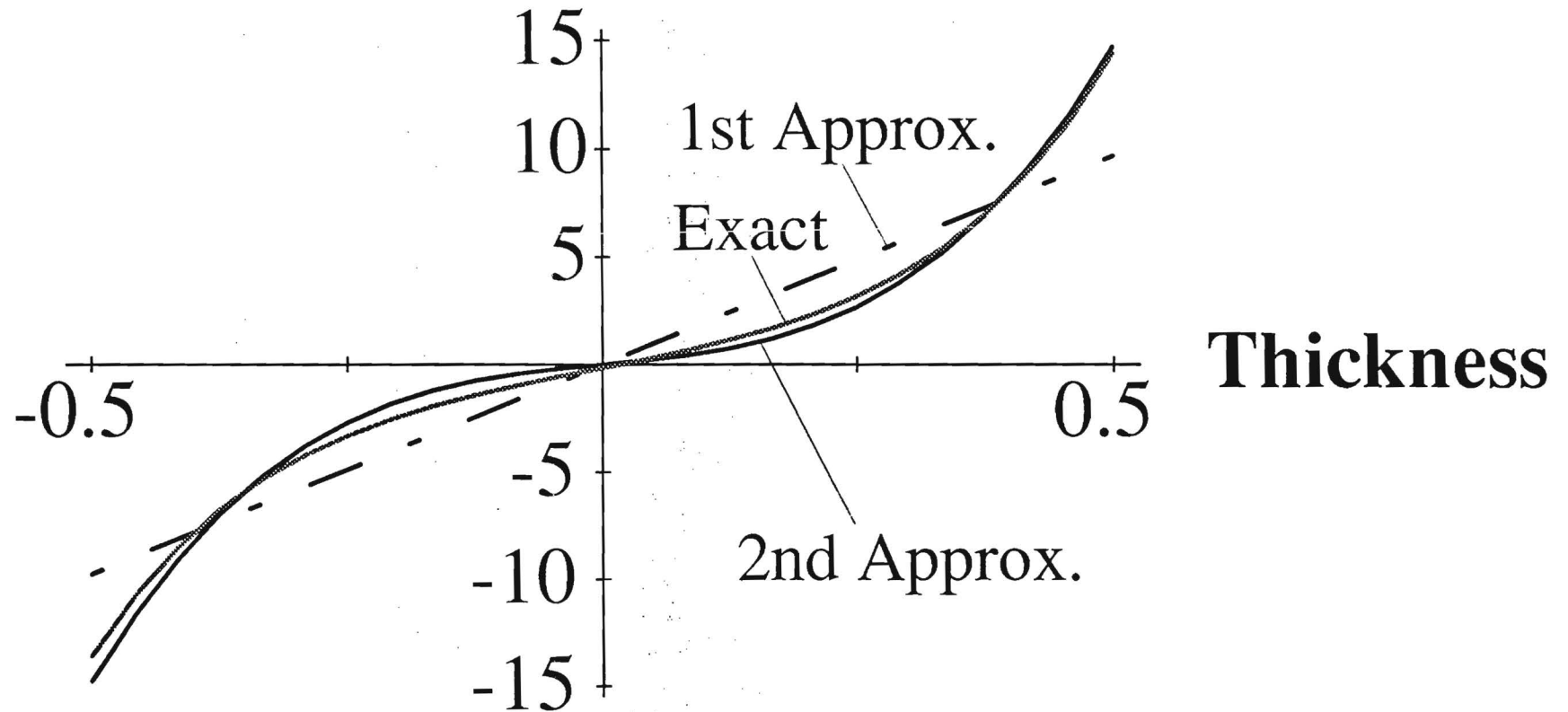


Fig. 2: Normalized in-plane stress distribution \bar{Z}_{11} versus normalized thickness coordinate.

Transverse Shear Stress at End

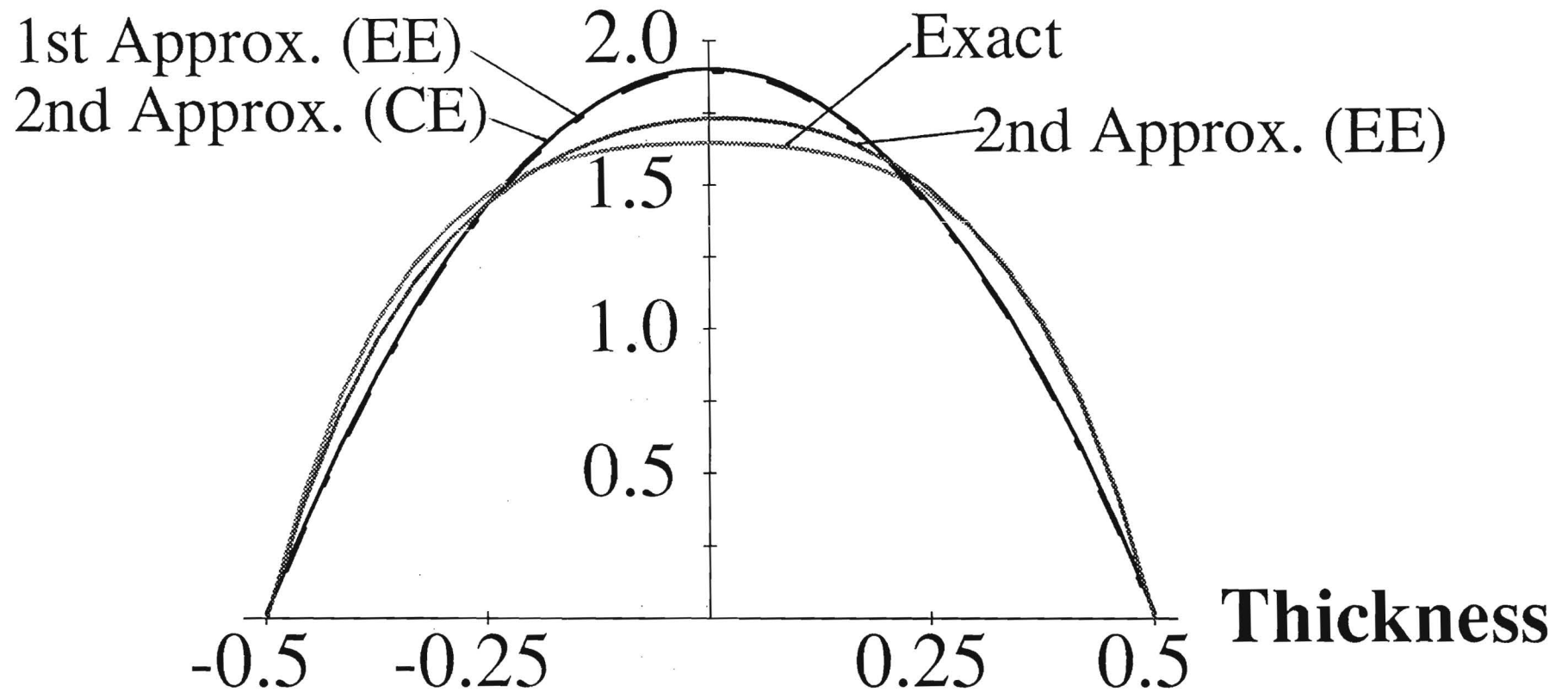


Fig. 3: Normalized transverse shear stress distribution $\bar{\tau}_{13}$ versus normalized thickness coordinate.

Transverse Normal Stress at L/2

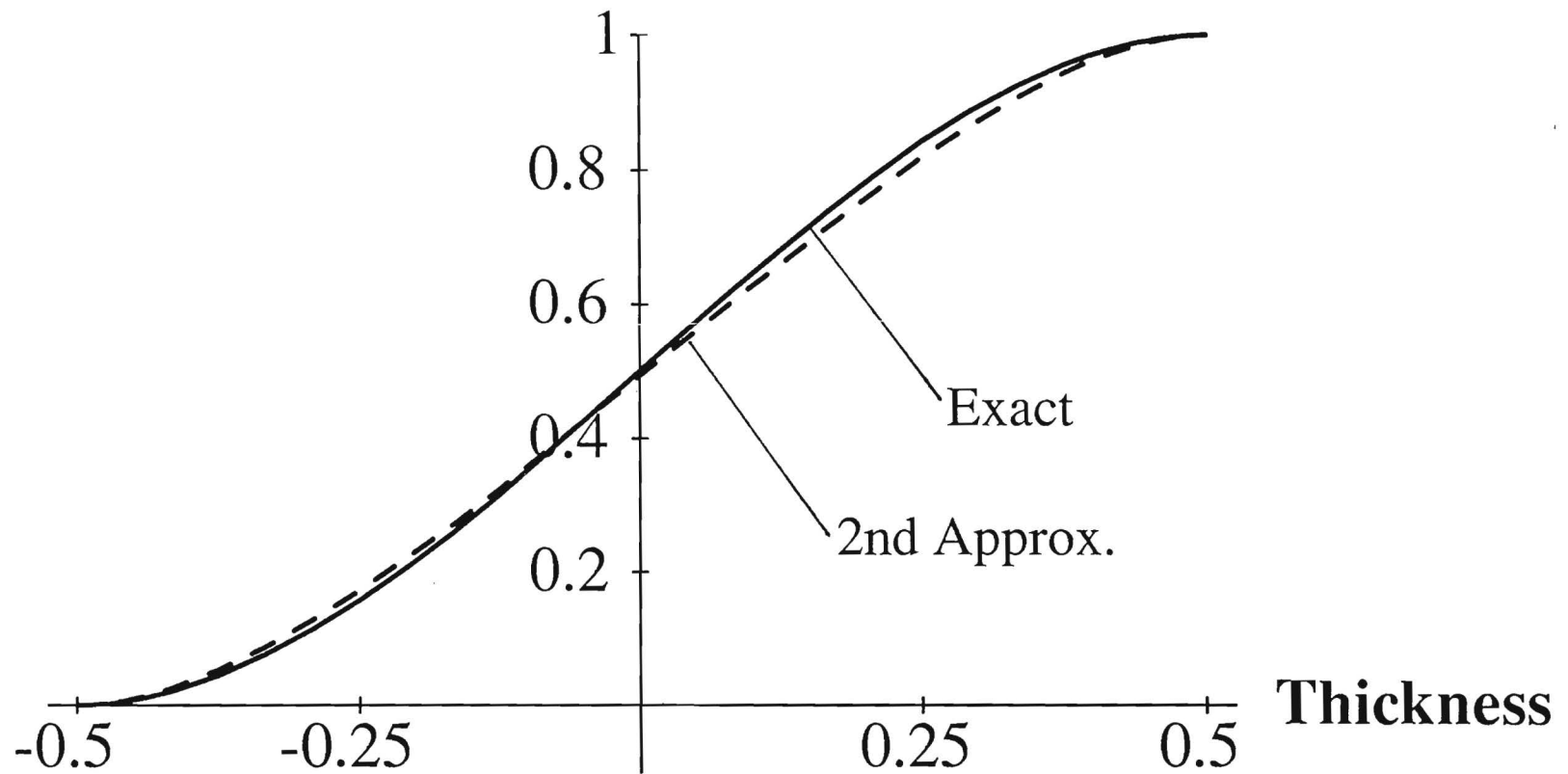


Fig. 4: Normalized transverse normal stress distribution \bar{Z}_{33} versus normalized thickness coordinate.

Modeling of Composite Beams and Plates for Static and Dynamic Analysis

**Interim Semi-Annual Report
NASA Grant NAG-1-1094
17 July 1991 – 16 January 1992**

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**Research Supported by
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Introduction

The purpose of this research is to develop a rigorous theory and corresponding computational algorithms for through-the-thickness analysis of composite plates.* This type of analysis is needed in order to find the elastic stiffness constants for a plate and to post-process the resulting plate solution in order to find approximate three-dimensional displacement, strain, and stress distributions throughout the plate.

In previous progress reports, a theoretical approach and attendant finite element method were described for the through-the-thickness analysis. Also, an analytical expression for the strain energy of nonhomogeneous, laminated composite plates was obtained by application of the variational-asymptotical method. We have shown that the first approximation of this analysis method corresponds exactly to the classical laminated plate theory, while the second approximation can be classified with a host of plate theories which account for shear deformation. No ad hoc shear correction factor is needed in our analysis; the shear stiffnesses are determined from closed-form expressions. However, we did show that $\frac{5}{6}$ is the appropriate shear correction factor with our theory for homogeneous plates with monoclinic symmetry about their mid-plane.

Finally, the geometrically nonlinear (possibly large deflection and rotation but with small strain) global deformation analysis of plates was addressed. This set of equations, along with the through-the-thickness analysis described above, comprise the complete boundary value problem for plates. We also looked briefly into the reasons why the drilling moment is such a difficult problem in plate analysis and its role in beam-plate connectivity. (This last item is really outside the scope of the present research program, and we still hope to develop a separately funded research program in this area.)

Summary of Work Done During This Period

Work proceeded at a somewhat lower level during the late Summer and early Fall because Mr. Lee was preparing for the Ph.D. qualifying examination, which he passed in November. We are still seeking a post doctoral fellow to help complete this work. Unless there are unforeseen difficulties, we anticipate the arrival of Dr. Sutyrin from Moscow within a few weeks.

Recall that our first publication on the dimensional reduction (via the variational-asymptotical method) was Atilgan and Hodges (1991), presented at the 1991 SDM meeting and submitted for publication at that time. While the paper was under review by the journal, we found that a term had been inadvertently dropped which would contribute to the second approximation if the plate possessed a certain type of nonhomogeneity through the thickness. Although we believed at first that this "troublesome term" was negligible,

* Secondly, it originally concerned prediction of the dynamic behavior of beams using finite elements in space-time. This secondary aspect of the work is being carried out under the primary funding of NSF, and all results obtained from that work will be made available to the Aerostructures Directorate.

early validation exercises seemed to indicate otherwise. Thus, our validation activity became very important to us: If we were able to correlate the present theory well with the exact solution, it would mean that the usual higher-degree polynomial expansions of the displacement and/or stress in many published higher-order theories, or the *large* numbers of extra variables in published layer-wise theories, would really be unnecessary. If not, our theory would have to be modified. The kinds of modifications of which we speak involve either altering the estimation scheme associated with the variational-asymptotical analysis or developing appropriate mathematical transformations which force the “troublesome term” to vanish. The latter entails introducing extra displacement variables, the number of which does not depend on the number of layers.

We are continuing to validate the through-the-thickness methodology described in Atilgan and Hodges (1992). We have focused on evaluating some results for laminated plates during this reporting period – especially to attempt to quantify its regime of applicability in terms of $\frac{h}{\ell}$. Also, we have developed a simplified “neo-classical theory,” described below, in which the transverse shear deformation appears in the *first* approximation and the “troublesome term” in a higher approximation. This should prove to be useful in fiber-reinforced composite materials for which the largest shear modulus is small relative to the largest extension modulus. Below we summarize the new analysis and the validation work to date as well as our publication activity.

Neo-Classical Analysis

In the first dimensional reduction strategy, found in Atilgan and Hodges (1992), the transverse shear strain measures turn out to be $O(\frac{h\varepsilon}{\ell})$ where ε is the order of the bending and extensional strain measures, h is the thickness of the plate, and ℓ is the wavelength of the deformation pattern. For that reason, they do not appear in the first approximation, which turns out to be identical to classical laminated plate theory. The transverse shear strains show up in the second approximation, and to keep the theory simple it is necessary to transform the reference surface so that derivatives of bending and extensional strain measures drop out of the strain energy. Unfortunately, this requires one to drop the above-mentioned “troublesome term.” Thus, the second approximation, while believed to be useful, is not asymptotically correct. It should be noted that this theory does not take into account the possibility that composites may be soft in shear which would result in the transverse shear strain measures being larger than they would be otherwise. Also, the theory is somewhat artificial in that boundary layer phenomena are not included although they are probably of at least as much importance as $\frac{h}{\ell}$ corrections.

Toward remedying this, we discovered through our work with composite beams that one can leave the transverse shear strains in the first approximation. This way, the first approximation contains more information than is strictly necessary for plates in which all the material constants are of the same order. Of course, this does not hurt anything. However, it buys us transverse shear deformability in a simpler theory. This could be of value when the shear moduli are small relative to the extension moduli, which is not uncommon in composites. Also, the boundary layer effects are of a higher order and thus need not be considered at this level. Finally, the “troublesome term” is in a higher

approximation as well. The new theory should be simpler than the old, but it should be almost as accurate.

Validation Activity

We have chosen to validate both the original dimensional reduction theory and the new neo-classical theory together. Results from both theories will be initially compared with exact, closed-form three-dimensional solutions for cylindrical bending from Pagano (1969), similar to the approach of Tessler and Saether (1990) and Cho (1991).

Cylindrical Bending Problem Let's consider an infinitely wide plate of thickness h and length L where the x_1, x_2, x_3 axes are aligned to the direction of length, infinite width, and thickness; respectively. The plate is simply supported at the ends (the x_1 extremities) and subject to sinusoidal upper surface loading.

The material properties are taken from Pagano (1969) and are given as

$$\begin{aligned} E_\ell &= 25 \times 10^6 \text{psi} & E_T &= 10^6 \text{psi} \\ G_{\ell T} &= 0.5 \times 10^6 \text{psi} & G_{TT} &= 0.2 \times 10^6 \text{psi} \\ \nu_{\ell T} &= \nu_{TT} = 0.25 \end{aligned}$$

where ℓ signifies the direction parallel to the fibers and T the transverse direction. The displacement, strain, and stress distributions through the thickness of a two-layer cross-ply laminate plate will be compared to the known exact elasticity solution, as described in Pagano (1969). This solution has been programmed in *Mathematica* so that detailed comparisons can be made for all quantities of interest.

Global Deformation Analysis (Plate Equations) Herein we will describe only the necessary terms for cylindrical bending from the general formulations reported previously. Notation follows that of our two papers described above. Kinematical equations can be reduced from Hodges *et al.* (1992) for the linear case as

$$\begin{aligned} \epsilon_{11} &= u_{1,1} \\ 2\epsilon_{12} &= u_{1,2} + u_{2,1} \\ \epsilon_{22} &= u_{2,2} \\ K_{11} &= \alpha_{1,1} \\ 2\kappa_{12} &= K_{12} + K_{21} = \alpha_{1,2} + \alpha_{2,1} \\ K_{22} &= \alpha_{2,2} \\ 2\gamma_{13} &= \alpha_1 + u_{3,1} \\ 2\gamma_{23} &= \alpha_2 + u_{3,2} \end{aligned} \tag{1}$$

Similarly, equilibrium equations are

$$\begin{aligned}
N_{11,1} + N_{12,2} + f_1 &= 0 \\
N_{12,1} + N_{22,2} + f_2 &= 0 \\
Q_{1,1} + Q_{2,2} + f_3 &= 0 \\
M_{11,1} + M_{12,2} - Q_1 &= 0 \\
M_{12,1} + M_{22,2} - Q_2 &= 0
\end{aligned} \tag{2}$$

Finally, putting the N 's in a column matrix called N , and similarly for all the other strain and stress resultants in Eqs. (1) and (2), one can write the constitutive laws for both the theory of Atılgan and Hodges (1992) and from the "neo-classical" theory in the same form. This can be expressed as

$$\begin{Bmatrix} N \\ M \\ Q \end{Bmatrix} = \begin{bmatrix} A & B & 0 \\ B^T & D & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \epsilon \\ K \\ \gamma \end{Bmatrix} \tag{3}$$

where A , B , and G are matrices of plate elastic constants obtained from the through-the-thickness analysis.

The plate is subject to sinusoidal surface loading of the form

$$f_3(x_1) = p_0 \sin\left(\frac{\pi x_1}{L}\right) \tag{4}$$

and $f_1 = f_2 = 0$.

Since the plate is infinite in the x_2 direction, all derivatives with respect to x_2 are zero. Plate boundary conditions for the simply supported case are

$$u_1(0) = N_{11}(L) = u_3(0) = u_3(L) = M_{11}(0) = M_{11}(L) = 0 \tag{5}$$

The triad \mathbf{B}_i can be expressed in terms of displacements and rotations with respect to the reference triad \mathbf{b}_i for cylindrical bending case. This is needed in order to develop three-dimensional expressions for the displacements. From specializing the equations of Hodges *et al.* (1992) for the linear case, we obtain

$$\begin{aligned}
\mathbf{B}_1 &= \mathbf{b}_1 + \frac{u_{2,1}}{2} \mathbf{b}_2 - \alpha_1 \mathbf{b}_3 \\
\mathbf{B}_2 &= -\frac{u_{2,1}}{2} \mathbf{b}_1 + \mathbf{b}_2 \\
\mathbf{B}_3 &= \alpha_1 \mathbf{b}_1 + \mathbf{b}_3
\end{aligned} \tag{6}$$

The plate problem is solved by first applying the resulting formulae of the one-dimensional through-the-thickness analysis to get the properties A , B , D , and G . Next, the above plate equations are solved for the given loading and boundary conditions. Finally, the resulting values of the two-dimensional strain measures (ϵ , K , and γ) are plugged into

the expressions for three-dimensional displacement, strain, and stress, which also result from the through-the-thickness analysis. Details of the through-the-thickness analyses are documented in Atilgan and Hodges (1992) and Hodges and Lee (1992).

Application of Through-the-Thickness Analysis During this reporting period we evaluated the displacement, strain, and stress distributions for 2-, 3-, and 4-layer cross-ply laminated plates using the second approximation of Atilgan and Hodges (1992) and the new neo-classical theory documented in Hodges and Lee (1992). These results were compared with the exact solution. In our validation work to date we have considered plates as thick as with $L/h=4$. Since this represents a relatively thick plate, the shear stresses play an important role in the behavior of the plate. Note that even for laminated plates, the solutions converge to the results of the first approximation, classical theory, as L/h increases. Namely, as the plate becomes thinner, the assumption of the normal element remaining straight and normal to the plate mid-surface becomes more reasonable. These results are being incorporated into Hodges and Lee (1992) and are not provided here.

Publication Activity

Our work towards deriving equations for the large deformation problem is described in a paper by Hodges *et al.* (1992), now accepted for publication in the *Journal of Applied Mechanics*. Abbreviated forms of this paper will be presented at SDM and SECTAM meetings in April this year. A copy of the final version will be transmitted to the technical monitor with this report.

Recall that our first publication on the dimensional reduction (via the variational-asymptotical method) was Atilgan and Hodges (1991), presented at the 1991 SDM meeting and submitted for publication at that time. After submission, it was found that a term had been inadvertently dropped which would contribute to the second approximation if the plate possessed a certain type of nonhomogeneity through the thickness. In the revised version the necessity to drop the "troublesome term" in order to obtain a strain energy of the simple form presented is discussed. That paper, Atilgan and Hodges (1992), has now been accepted for publication in the *International Journal of Solids and Structures*. A copy of the final version will be sent to the technical monitor by the end of the month.

As discussed above, a new dimensional reduction, based on a slightly simpler estimation scheme, has now been formulated. Hodges and Lee (1992) contains the details of this neo-classical analysis along with validation results for both it and the original theory. The final version of this paper will be sent to the technical monitor in early February.

Future work

In the balance of the grant we intend to:

1. continue validation of the analytical solution for symmetric, anti-symmetric, and general (asymmetric) laminated plates;
2. develop a modeling analysis for layered generally anisotropic plates;

3. and either develop or obtain a finite element analysis for global deformation of plates.

In the long term, and under a new project, we still tentatively plan to:

1. look into the boundary energy of plates;
2. work toward fast, robust, and accurate computational techniques for the complete plate problem, including specifically the treatment of plate/beam connections and plate/plate connections at oblique angles;
3. and extend the dimensional reduction scheme to apply to laminated shells.

References

Atilgan, Ali R.; and Hodges, Dewey H. (1991): On the Strain Energy of Laminated Composite Plates. *Proceedings of the 32nd Structures, Structural Dynamics and Materials Conference*, Baltimore, Maryland, April 8 – 10, pp. 996 – 1007 (AIAA Paper 91-0960).

Atilgan, Ali R.; and Hodges, Dewey H. (1992): On the Strain Energy of Laminated Composite Plates. *International Journal of Solids and Structures*, to appear.

Cho, M. (1991): An Efficient Higher-Order Composite Plate Theory for General Lamination Configurations. *Proceedings of the 32nd Structures, Structural Dynamics and Materials Conference*, Baltimore, Maryland, April 8 – 10, pp. 1496 – 1510 (AIAA Paper 91-1204).

Hodges, Dewey H.; Atilgan, Ali R.; and Danielson, Donald A. (1992): A Geometrically Nonlinear Theory of Elastic Plates. *Journal of Applied Mechanics*, to appear.

Hodges, Dewey H.; and Lee, Bok W. (1992): Application of the Variational-Asymptotical Method to Composite Plates. 33rd Structures, Structural Dynamics and Materials Conference, Dallas, Texas, April 13 – 15, in preparation.

Pagano, N. J. (1969): Exact Solutions for Composite Laminates in Cylindrical Bending. *Journal of Composite Materials*, Vol. 3, No. 7, pp. 397 – 411.

Tessler, A.; and Saether, E. (1990): A Computationally Viable Higher-order Theory for Laminated Composite Plates. *International Journal for Numerical Methods in Engineering*, Vol. 31, pp. 1069 – 1086.

**Modeling of Composite Beams and Plates
for Static and Dynamic Analysis**

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Summary of Progress Prior to This Period

The main purpose of this research has been to develop a rigorous theory and corresponding computational algorithms for through-the-thickness analysis of composite plates. This type of analysis is needed in order to find the elastic stiffness constants for a plate and to post-process the resulting plate solution in order to find approximate three-dimensional displacement, strain, and stress distributions throughout the plate.

We have settled on the variational-asymptotical method (VAM)¹ as a suitable framework in which to solve these types of problems. The VAM was applied to laminated plates with constant thickness in the work of Atilgan and Hodges². The corresponding geometrically nonlinear global deformation analysis of plates was developed by Hodges, Atilgan, and Danielson³. A different application of VAM, along with numerical results, was obtained by Hodges, Lee, and Atilgan⁴. (Copies of these papers have been delivered to Mr. Hinnant.)

In Ref. 2, the "first approximation" is exactly the same as classical laminated plate theory. The "second approximation" takes transverse shear deformation into account and is developed only for plates with certain restrictions in their construction. To remove the restrictions one must "kill" certain interaction terms in the strain energy, and the means for doing so for general laminated plates were not given in that paper.

In Ref. 3, a set of kinematical and intrinsic equilibrium equations are derived for large deflection and rotation but with small strain. The relationship between the drilling rotation and the other kinematical variables gives new insight into the drilling moment and its role in beam-plate connectivity. This work has shown that drilling type rotation is not an independent degree of freedom in plate theory. An applied drilling moment at a point on a plate is not resisted at all by the plate. Such a moment, in order to have any physical resistance from the plate, must be applied over a finite area. Other than this, a point drilling moment can only be resisted by a plate if the plate model is derived from couple-stress elasticity⁵.

The development in Ref. 4 includes transverse shear in the "first approximation" and is stopped there. Results from this theory were compared, for the cylindrical bending case, with results from the exact solution of Pagano⁶ for cross-ply laminated plates. The resulting theory, termed a "neo-classical" theory, is at least as good as classical theory in every case and for some cases superior to it. Further work was judged to be needed in order to correlate with shear-coupled laminates, also treated by Pagano⁷.

Summary of Work Done During This Period

Work during this reporting period has continued along two lines: (1) We have continued to evaluate the neo-classical plate theory (NCPT) for the shear-coupled laminates and (2) we began to explore, with permission from the technical monitor, what kinds of considerations would be involved to model plates with nonuniform thickness, which has led to considerable progress toward development of higher approximations of our constant thickness plate models.

Evaluation of NCPT for Shear-Coupled Laminates

Upon finishing the validation of NCPT for bidirectional plates, we continued to evaluate the response of plates with arbitrary stacking sequences. This is a rather challenging problem. Since the fiber orientations are not parallel to the axis of each laminate, the influence of shear-coupling becomes more evident for this type of laminated plates. When the fiber orientation coincides with the principal elastic axes of each laminate no shear-coupling terms exists. For validating the

theory, example cases are chosen from Pagano⁷, in which the exact, closed-form three-dimensional solutions are available. The results shows that the new theory (NCPT) is more accurate than classical laminated plate theory (CPT) when thickness of plate increases. Since NCPT has more kinematical variables than CPT, we expect it to be more accurate for thick plates.

After summarizing the through-the-thickness analysis, we will present the governing equations for the global plate analysis with arbitrary stacking sequences. Finally, we will present the solutions.

Three-Dimensional Description Consider a plate of constant thickness h composed of layers, each of which is homogeneous and possesses monoclinic material symmetry about its mid-plane; a schematic of the plate mid-surface is shown in Fig. 1. Let us introduce Cartesian coordinates x_i so that x_α denotes lengths along orthogonal straight lines in the mid-surface of the undeformed plate, and $x_3 = h\zeta$ is the distance of an arbitrary point to the mid-surface in the undeformed plate, where $-\frac{1}{2} \leq \zeta \leq \frac{1}{2}$. Throughout the analysis, Greek indices assume values 1 or 2; Latin indices assume values 1, 2, and 3; and repeated indices are summed over their ranges.

Now, letting \mathbf{b}_i denote an orthogonal reference triad along the undeformed plate coordinate lines, one can express the position vector from a fixed point O to an arbitrary point as

$$\hat{\mathbf{r}}(x_1, x_2, \zeta) = x_\alpha \mathbf{b}_\alpha + h\zeta \mathbf{b}_3 = \mathbf{r}(x_1, x_2) + h\zeta \mathbf{b}_3 \quad (1)$$

The position vector to the mid-surface is also the average position of points along the normal line, at a particular value of x_1 and x_2 , so that

$$\mathbf{r} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{\mathbf{r}} d\zeta = \langle \hat{\mathbf{r}} \rangle \quad (2)$$

The angle brackets $\langle \rangle$ are used throughout the paper to denote the integral through the thickness.

Now, in accordance with Ref. 3 the position vector of any point in the deformed plate is

$$\begin{aligned} \hat{\mathbf{R}}(x_1, x_2, \zeta) = & \mathbf{R}(x_1, x_2) + h\zeta \mathbf{B}_3(x_1, x_2) \\ & + hw_i(x_1, x_2, \zeta) \mathbf{B}_i(x_1, x_2) \end{aligned} \quad (3)$$

where \mathbf{R} is defined as

$$\begin{aligned} \mathbf{R}(x_1, x_2) = & \langle \hat{\mathbf{R}}(x_1, x_2, \zeta) \rangle \\ = & \mathbf{r}(x_1, x_2) + \mathbf{u}(x_1, x_2) \end{aligned} \quad (4)$$

and where \mathbf{u} is the plate displacement vector, defined as the position vector from a point on the undeformed plate mid-surface to the corresponding point on the average surface of the deformed plate. The \mathbf{B}_i triad is defined so that

$$\mathbf{B}_1 \cdot \mathbf{R}_{,2} = \mathbf{B}_2 \cdot \mathbf{R}_{,1} \quad (5)$$

and $\mathbf{B}_3 = \mathbf{B}_1 \times \mathbf{B}_2$ is parallel to $\langle \zeta \hat{\mathbf{R}} \rangle$. These definitions give rise to the same kinematical constraints on the warping as suggested by Hodges *et al.*³

$$\langle w_i \rangle = 0 \quad \langle \zeta w_\alpha \rangle = 0 \quad (6)$$

We now turn to the strain field, details of which can be found in Ref. 2. First we arrange the six strain components into a matrix form so that

$$\Gamma = [\Gamma_e \quad 2\Gamma_s \quad \Gamma_t]^T \quad (7)$$

where Γ_e includes the extensional and in-plane shearing strains, and Γ_s and Γ_t contains the transverse shear and transverse normal strains, respectively. Thus,

$$\begin{aligned} \Gamma_e &= [\Gamma_{11} \quad 2\Gamma_{12} \quad \Gamma_{22}]^T \\ 2\Gamma_s &= [2\Gamma_{13} \quad 2\Gamma_{23}]^T \\ \Gamma_t &= \Gamma_{33} \end{aligned} \quad (8)$$

Next, for notational convenience, we introduce the matrix operators

$$\partial_t = h \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{bmatrix} \quad \partial_e = h \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \\ 0 & \frac{\partial}{\partial x_2} \end{bmatrix} \quad (9)$$

and generalized strain measures in matrix form given by

$$\begin{aligned} \epsilon &= [\epsilon_{11} \quad 2\epsilon_{12} \quad \epsilon_{22}]^T \\ K &= [K_{11} \quad 2K_{12} \quad K_{22}]^T \\ 2\gamma &= [2\gamma_{13} \quad 2\gamma_{23}]^T \end{aligned} \quad (10)$$

where ϵ_{11} and ϵ_{22} are the plate extensional strain measures, $2\epsilon_{12}$ is the plate in-plane shear strain measure, $2\gamma_{\alpha 3}$ are the plate transverse shear strain measures, K_{11} and K_{22} are the plate bending measures, and $2\kappa_{12} = K_{12} + K_{21}$ is the plate twisting measure. All these measures are functions only of x_1 and x_2 , and their explicit forms for large deformation are given in Ref. 3.

Denoting the in-plane warping by

$$w_{\parallel} = [w_1 \quad w_2]^T \quad (11)$$

one can now write the Jaumann strain components following Ref. 3 as

$$\begin{aligned} \Gamma_e &= \epsilon + \zeta K h + \partial_e w_{\parallel} \\ 2\Gamma_s &= w'_{\parallel} + 2\gamma + \partial_t w_3 \\ \Gamma_t &= w'_3 \end{aligned} \quad (12)$$

where $()'$ denotes the derivative with respect to ζ .

A similar procedure can be followed for the conjugate stresses so that

$$\begin{aligned} Z_e &= [Z_{11} \quad Z_{12} \quad Z_{22}]^T \\ Z_s &= [Z_{13} \quad Z_{23}]^T \\ Z_t &= Z_{33} \end{aligned} \quad (13)$$

where Z_e contains the extensional and in-plane shear stresses while Z_s and Z_t have the transverse shear and transverse normal stresses. The stress components may then be written in a matrix form as

$$Z = [Z_e \quad Z_s \quad Z_t]^T \quad (14)$$

In light of this, the three-dimensional constitutive law can be expressed as

$$\begin{Bmatrix} Z_e \\ Z_s \\ Z_t \end{Bmatrix} = \begin{bmatrix} D_e & D_{es} & D_{et} \\ D_{es}^T & D_s & D_{st} \\ D_{et}^T & D_{st}^T & D_t \end{bmatrix} \begin{Bmatrix} \Gamma_e \\ 2\Gamma_s \\ \Gamma_t \end{Bmatrix} \quad (15)$$

where D_e , D_{es} , D_{et} , D_s , D_{st} , and D_t are 3×3 , 3×2 , 3×1 , 2×2 , 2×1 , and 1×1 matrices, respectively. Here, this law is written for directions parallel to plate coordinate axes, which are not in general along the material axis. Therefore, the material constants, D 's, are the transformed values from material axis to the plate axis.

The plate strain energy per unit area can then be written as

$$J = \frac{1}{2} \langle Z^T \Gamma \rangle \quad (16)$$

Following Ref. 2, we decompose the strain energy into two positive definite, quadratic forms. Here we define the extensional strain energy J_{\parallel} , and the transverse strain energy J_{\perp} (containing contributions from both transverse normal and shear strains) as

$$\begin{aligned} J_{\parallel} &= \min_{\Gamma_s, \Gamma_t} J \\ J_{\perp} &= J - J_{\parallel} \end{aligned} \quad (17)$$

When the material fiber direction is oriented parallel to the plane of the plate, such that each lamina exhibits a monoclinic symmetry, D_{es} and D_{st} will vanish. In this case the extensional and transverse energies can be written in terms of the three-dimensional material properties in the following simple form

$$\begin{aligned} 2J_{\parallel} &= \langle \Gamma_e^T D_{\parallel} \Gamma_e \rangle \\ 2J_{\perp} &= \langle 2\Gamma_s^T D_s 2\Gamma_s + D_t (\Gamma_t + D_{\perp} \Gamma_e)^2 \rangle \end{aligned} \quad (18)$$

where

$$D_{\parallel} = D_e - D_{et} D_{\perp} \quad D_{\perp} = D_t^{-1} D_{et}^T \quad (19)$$

This completes the three-dimensional description of the displacement, strain, and stress fields. These three-dimensional fields are not really suitable for plate analysis because of the three-dimensional warping variables w_i . We now turn to elimination of the warping by dimensional reduction.

Dimensional Reduction In the following sections, we will apply the variational-asymptotical method of Ref. 1 for nonhomogeneous, laminated plates in pursuit of a first approximation of the plate strain energy. Before doing so, however, it is appropriate to discuss the estimation procedure. First, we introduce upper bounds on the in-plane, bending, and transverse shear strain measures ϵ_e , ϵ_b , and ϵ_s , respectively, such that

$$\sqrt{\epsilon^T \epsilon} \leq \epsilon_e \quad \frac{h}{2} \sqrt{K^T K} \leq \epsilon_b \quad \sqrt{2\gamma^T 2\gamma} \leq \epsilon_s \quad (20)$$

For the first approximation we need only to keep terms in the strain field that are of the order of ε where

$$\varepsilon_e + \varepsilon_b + \varepsilon_s \leq \varepsilon \quad (21)$$

This implies that we will have strain energy density of the order $\mu\varepsilon^2$ where μ is of the order of the elastic moduli.

In order to take advantage of the physical aspects of plate deformation, we introduce another small parameter h/ℓ , where ℓ is the smallest constant for which all of the following hold for all possible combinations of α and β

$$\sqrt{\varepsilon_{,\alpha}^T \varepsilon_{,\beta}} \leq \frac{\varepsilon_e}{\ell} \quad \frac{h}{2} \sqrt{K_{,\alpha}^T K_{,\beta}} \leq \frac{\varepsilon_b}{\ell} \quad \sqrt{2\gamma_{,\alpha}^T 2\gamma_{,\beta}} \leq \frac{\varepsilon_s}{\ell} \quad (22)$$

One may think of ℓ as the wavelength of the deformation pattern.

Rather than write out complete expressions for the strain field, we will only write the terms needed for the first approximation. By consideration of the above set of estimation parameters, the strain expression can be approximated as

$$\begin{aligned} \Gamma_e &= \epsilon + \zeta K h \\ 2\Gamma_s &= 2\gamma + w'_{\parallel} \\ \Gamma_t &= w'_3 \end{aligned} \quad (23)$$

Thus, the warping is not present in extensional energy, and the only part of the strain energy remaining to be minimized is the transverse energy

$$\begin{aligned} 2J_{\perp} &= \left\langle D_t [w'_3 + D_{\perp} (\epsilon + \zeta K h)]^2 \right\rangle \\ &+ \left\langle (2\gamma + w'_{\parallel})^T D_s (2\gamma + w'_{\parallel}) \right\rangle \end{aligned} \quad (24)$$

The variational-asymptotical method calls for the minimization of this strain energy expression with respect to the warping, with the constraints given in Eq. (6), resulting in

$$\begin{aligned} w_3 &= D_{\perp 1} \epsilon + D_{\perp 2} K h \\ w_{\parallel} &= \left[\frac{1}{12} (H_{s1} - 8H_{s2}) (\langle \zeta H_{s1} \rangle - 8\langle \zeta H_{s2} \rangle)^{-1} - I_2 \zeta \right] 2\gamma \end{aligned} \quad (25)$$

in which I_2 is the 2×2 identity matrix and

$$\begin{aligned} D'_{\perp 1} &= -D_{\perp} & D'_{\perp 2} &= -\zeta D_{\perp} \\ H'_{s1} &= D_s^{-1} & H'_{s2} &= \frac{\zeta^2}{2} D_s^{-1} \end{aligned} \quad (26)$$

These expressions for the warping are determined uniquely by imposing the continuity of $D_{\perp\alpha}$ and $H_{s\alpha}$ between the layers together with constraints Eq. (6) so that

$$\langle D_{\perp\alpha} \rangle = \langle H_{s\alpha} \rangle = 0 \quad (27)$$

for out-of-plane and in-plane warping, respectively. These constraints guarantee that warping functions are continuous. However, strain and stress, which are functions of the derivatives of the warping, may or may not be continuous.

The strain energy per unit area of the plate is obtained by substituting the warping from Eq. (25) into the strain energy, yielding

$$J = \frac{1}{2} [\epsilon^T (A\epsilon + 2BK) + K^T DK + 2\gamma^T G2\gamma] \quad (28)$$

where

$$\begin{aligned} A &= h\langle D_{\parallel} \rangle & B &= h^2\langle \zeta D_{\parallel} \rangle \\ D &= h^3\langle \zeta^2 D_{\parallel} \rangle & G &= h\langle g^T D_s g \rangle \end{aligned} \quad (29)$$

and where

$$g = \frac{1}{12} (1 - 4\zeta^2) D_s^{-1} (\langle \zeta H_{s1} \rangle - 8\langle \zeta H_{s2} \rangle)^{-1} \quad (30)$$

Note that D_{\parallel} corresponds to \bar{Q} , the well known transformed reduced stiffness matrix from classical laminated plate theory⁸. It is possible to define the force, moment and transverse shear stress resultants N , M , and Q respectively, as

$$\begin{aligned} N &= h\langle Z_e \rangle = \left(\frac{\partial J}{\partial \epsilon} \right)^T \\ M &= h^2\langle \zeta Z_e \rangle = \left(\frac{\partial J}{\partial K} \right)^T \\ Q &= h\langle Z_s \rangle = \left[\frac{\partial J}{\partial (2\gamma)} \right]^T \end{aligned} \quad (31)$$

Then, based on the strain energy, the plate constitutive law can be expressed as

$$\begin{Bmatrix} N \\ M \\ Q \end{Bmatrix} = \begin{bmatrix} A & B & 0 \\ B^T & D & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \epsilon \\ K \\ 2\gamma \end{Bmatrix} \quad (32)$$

Note that transverse normal stress is zero in this theory, and thus, we should not expect the transverse normal strain to be very accurate from Eqs. (12) unless we extend the theory to higher approximations. Similarly, we do not expect the transverse shear stress and strain to be very accurate with the present theory, since some important contributions to its detailed variation are associated with h/ℓ corrections to the energy¹. These quantities can be obtained from integrating the three-dimensional equilibrium equations through the thickness to get the transverse shear and normal stresses and applying the three-dimensional constitutive law to get the transverse shear and normal strains.

This concludes the dimensional reduction. The global deformation equations³, along with these plate constitutive equations, comprise what we term the neo-classical theory. (Note that the displacement shift mentioned in Refs. 2 and 3 is not necessary with this theory, and thus $2\gamma^* = 2\gamma$.) With the warping known in terms of ϵ , K , and 2γ , which in turn are known through solution of the global deformation problem, it is now possible to evaluate three-dimensional approximations of

displacement, strain, and stress fields. For the purpose of validating the stiffness model and field relations, however, only a specialized version of the global deformation analysis is undertaken here.

Linear Plate Equations The global deformation equations that correspond to the above strain energy function were developed in Ref. 3. Since ϵ , K , and 2γ are nonlinear functions of the displacement and rotation variables, this theory is applicable to large deformation of plates, and these can now be obtained by solving a specific problem. Here, rather than repeat the entire formulation, we will specialize it for linear, cylindrical bending problems, which we will use below for validation of the dimensional reduction scheme of the previous section.

Kinematical equations from Ref. 3 for the linear case are given as

$$\begin{aligned}
\epsilon_{11} &= u_{1,1} \\
2\epsilon_{12} &= u_{1,2} + u_{2,1} \\
\epsilon_{22} &= u_{2,2} \\
K_{11} &= \theta_{1,1} \\
2\kappa_{12} &= \theta_{1,2} + \theta_{2,1} \\
K_{22} &= \theta_{2,2} \\
2\gamma_{13} &= \theta_1 + u_{3,1} \\
2\gamma_{23} &= \theta_2 + u_{3,2}
\end{aligned} \tag{33}$$

where $u_i = \mathbf{u} \cdot \mathbf{b}_i$ and $\theta_\alpha = \mathbf{B}_3 \cdot \mathbf{b}_\alpha$.

Similarly, equilibrium equations are

$$\begin{aligned}
N_{11,1} + N_{12,2} + f_1 &= 0 \\
N_{12,1} + N_{22,2} + f_2 &= 0 \\
Q_{1,1} + Q_{2,2} + f_3 &= 0 \\
M_{11,1} + M_{12,2} - Q_1 &= 0 \\
M_{12,1} + M_{22,2} - Q_2 &= 0
\end{aligned} \tag{34}$$

where f_i are the applied loads per unit area of the plate.

Cylindrical Bending Analysis Consider a plate of length L along x_1 and infinite width in the x_2 direction shown in Fig. 2. All derivatives with respect to x_2 are zero which causes many of the variables in the above equilibrium, constitutive, and kinematical equations to drop out.

In the case of cylindrical bending, the plate is subject to sinusoidal surface loading of the form

$$f_3(x_1) = p_0 \sin(px_1) \tag{35}$$

where $p = \frac{\pi}{L}$ and $f_\alpha = 0$. For the three-dimensional analysis, we assume the loading f_3 to be imposed in the form of an upper surface traction.

We consider a simply supported plate, the boundary conditions of which are

$$u_3(0) = u_3(L) = 0 \quad u_{\alpha,1}(0) = u_{\alpha,1}(L) = 0 \quad \theta_{\alpha,1}(0) = \theta_{\alpha,1}(L) = 0 \tag{36}$$

The governing system of equations can be put into a matrix form by defining generalized coordinates, strain measures, stress and moment results, and loading. Let

$$q = [u_1 \quad u_2 \quad \theta_1 \quad \theta_2 \quad u_3]^T \quad (37)$$

$$\varepsilon = [\epsilon_{11} \quad 2\epsilon_{12} \quad \kappa_{11} \quad 2\kappa_{12} \quad 2\gamma_{13} \quad 2\gamma_{23}]^T \quad (38)$$

$$F = [N_{11} \quad N_{12} \quad M_{11} \quad M_{12} \quad Q_1 \quad Q_2]^T \quad (39)$$

$$f = [0 \quad 0 \quad 0 \quad 0 \quad -f_3]^T \quad (40)$$

Eqs. (33) can be written as

$$\varepsilon = \mathcal{D} q \quad (41)$$

where

$$\mathcal{D} = \begin{bmatrix} \partial & 0 & 0 & 0 & 0 \\ 0 & \partial & 0 & 0 & 0 \\ 0 & 0 & \partial & 0 & 0 \\ 0 & 0 & 0 & \partial & 0 \\ 0 & 0 & 1 & 0 & \partial \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \partial & 0 & 0 & 0 & 0 \\ 0 & \partial & 0 & 0 & 0 \\ 0 & 0 & \partial & 0 & 0 \\ 0 & 0 & 0 & \partial & 0 \\ 0 & 0 & 0 & 0 & \partial \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \partial^* + k \quad (42)$$

in which ∂ denotes the derivative with respect to x_1 . Eqs. (34) can be written as

$$f = \mathcal{E} F \quad (43)$$

where

$$\mathcal{E} = \begin{bmatrix} \partial & 0 & 0 & 0 & 0 & 0 \\ 0 & \partial & 0 & 0 & 0 & 0 \\ 0 & 0 & \partial & 0 & -1 & 0 \\ 0 & 0 & 0 & \partial & 0 & -1 \\ 0 & 0 & 1 & 0 & \partial & 0 \end{bmatrix} = \partial^{*T} - k^T \quad (44)$$

Now plate constitutive equation can be written as

$$F = K^* \varepsilon \quad (45)$$

where

$$K^* = \begin{bmatrix} A^* & B^* & 0 \\ B^* & D^* & 0 \\ 0 & 0 & G \end{bmatrix} \quad (46)$$

Here the starred matrices are 2×2 sub-matrices consisting of the first 2 rows and columns of the corresponding matrices in Eqs. (32), resulting in K^* being a 6×6 matrix. So the given problem

yields 17 equations in terms of 17 unknowns. Combining Eqs. (41), (43), and (45), the equilibrium equations can be written as

$$(\partial^{*T} - k^T) K^* (\partial^* + k) = f \quad (47)$$

which can be put into the form

$$M_c q_{,11} + C_c q_{,1} - K_c q = f \quad (48)$$

where

$$M_c = \begin{bmatrix} A_{11}^* & A_{12}^* & B_{11}^* & B_{12}^* & 0 \\ A_{12}^* & A_{22}^* & B_{12}^* & B_{22}^* & 0 \\ B_{11}^* & B_{12}^* & D_{11}^* & D_{12}^* & 0 \\ B_{12}^* & B_{22}^* & D_{12}^* & D_{22}^* & 0 \\ 0 & 0 & 0 & 0 & G_{11} \end{bmatrix} = \begin{bmatrix} A^* & B^* & 0 \\ B^* & D^* & 0 \\ 0 & 0 & G_{11} \end{bmatrix} \quad (49)$$

$$C_c = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -G_{11} \\ 0 & 0 & 0 & 0 & -G_{12} \\ 0 & 0 & G_{11} & G_{12} & 0 \end{bmatrix} \quad (50)$$

$$K_c = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & G_{11} & G_{12} & 0 \\ 0 & 0 & G_{12} & G_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (51)$$

The above equation can be solved directly with associated boundary conditions. But let us take advantage of the sparsity of matrices in the equation. We can break down this equation as follows

$$A^* u_{,11} + B^* \theta_{,11} = 0 \quad (52)$$

$$B^* u_{,11} + D^* \theta_{,11} - [G_{11} \ G_{12}]^T u_{3,1} - G \theta = 0 \quad (53)$$

$$G_{11} u_{3,11} + [G_{11} \ G_{12}] \theta = -f_3 \quad (54)$$

when u and θ are arranged as

$$u = [u_1 \ u_2]^T \quad \theta = [\theta_1 \ \theta_2]^T \quad (55)$$

Since f_3 is a simple sinusoidal function given in Eq. (35), θ can be uncoupled by putting Eqs. (52) and (54) into Eq. (53) yielding

$$M_c^* \theta_{,11} + K_c^* \theta = f_c^* \frac{p_0}{p} \cos(px_1) \quad (56)$$

where

$$M_c^* = D^* - B^* A^{*-1} B^* \quad K_c^* = \begin{bmatrix} 0 & 0 \\ 0 & \frac{G_{12}^2}{G_{11}} - G_{22} \end{bmatrix} \quad f_c^* = [1 \ \frac{G_{12}}{G_{11}}]^T$$

Since M_c^* is symmetric, positive definite and K_c^* symmetric, Eq. (56) can be decoupled as is. Note that when plate is symmetrically layered up B^* matrix vanishes, resulting $M_c^* = D^*$.

Now, introducing the coordinate transformation

$$\theta = \phi y \quad (57)$$

where ϕ is a eigenvector matrix, we diagonalize Eq. (56) which gives

$$M_d y_{,11} + K_d y = f_d \frac{p_0}{p} \cos(px_1) \quad (58)$$

where

$$M_d = \phi^T M_c^* \phi \quad K_d = \phi^T K_c^* \phi \quad f_d = \phi^T f_c^* \quad (59)$$

By solving above decoupled equations for y , we can recover original solution θ using Eq. (57). It is easy to shown that all integration constants produced in the calculation vanish for cylindrical bending. In other words, we need only to obtain particular solution for Eq. (56) or Eq. (58). Note that since effects of boundary layer belong to higher order for this type of first approximation, we can neglect a homogeneous solution in the first place.

After some calculation, θ can be found. Also u_3 and u are determined by putting θ into Eqs. (52) and Eqs. (54). Rotations, in-plane displacements, and out-of-plane displacement are obtained as follows

$$\theta = \phi [K_d - p^2 M_d]^{-1} f_d \frac{p_0}{p} \cos(px_1) \quad (60)$$

$$u = -A^{*-1} B^* [K_d - p^2 M_d]^{-1} f_d \frac{p_0}{p} \cos(px_1) \quad (61)$$

$$u_3 = \frac{1}{G_{11}} \left(1 - [G_{11} \quad G_{12}] \phi [K_d - p^2 M_d]^{-1} f_d \right) \frac{p_0}{p^2} \sin(px_1) \quad (62)$$

To recover three-dimensional fields, one can substitute values obtained for ϵ , K , and 2γ by taking derivatives of above expressions into Eqs. (25), (23), and (15), to get warping, strain, and stress, respectively. Note that all the strain measures can be calculated to order ϵ by using Eq. (23); however, Γ_e can be determined to order $h\epsilon/\ell$ by use of Eq. (12). To avoid inconsistency, one should use Eq. (23) for all strain components when calculating the stress components. However, NCPT is more accurate than CPT since it picked up the general strain measures which do not appear in CPT. In other words, the extra strain measures reflect the effects of transverse shear deformation, resulting in a better solution.

For the purpose of comparing the displacement field with that of linear, three-dimensional elasticity, we need to determine the \mathbf{b}_i measure numbers of the displacement field

$$z_i = \mathbf{b}_i \cdot (\hat{\mathbf{R}} - \hat{\mathbf{r}}) \quad (63)$$

With the warping known from Eqs. (25) in terms of ϵ , K , and 2γ , and the triad \mathbf{B}_i given as

$$\begin{aligned} \mathbf{B}_1 &= \mathbf{b}_1 + \frac{u_{2,1}}{2} \mathbf{b}_2 - \theta_1 \mathbf{b}_3 \\ \mathbf{B}_2 &= -\frac{u_{2,1}}{2} \mathbf{b}_1 + \mathbf{b}_2 - \theta_2 \mathbf{b}_3 \\ \mathbf{B}_3 &= \theta_1 \mathbf{b}_1 - \theta_2 \mathbf{b}_2 + \mathbf{b}_3 \end{aligned} \quad (64)$$

The functions z_i can now be expressed in terms of global deformation variables. The result is that

$$\begin{aligned} z_1 &= u_1 + h\zeta\theta_1 + hw_1 \\ z_2 &= u_2 - h\zeta\theta_2 + hw_2 \\ z_3 &= u_3 + hw_3 \end{aligned} \quad (65)$$

where u_i , θ_α are obtainable in Eqs. (60), (61), and (62), and w_i in Eq. (25).

Results The intent of this section is to compare results from the exact solution with results from the present theory. The exact three-dimensional elasticity solution for cylindrical bending of angle ply laminated plates was obtained by Pagano⁷, and the results presented herein labelled as "exact" generated from his equations. The material properties are⁷

$$\begin{aligned} E_L &= 25 \times 10^6 \text{psi} & E_T &= 10^6 \text{psi} \\ G_{LT} &= 0.5 \times 10^6 \text{psi} & G_{TT} &= 0.2 \times 10^6 \text{psi} \\ \nu_{LT} &= \nu_{TT} = 0.25 \end{aligned} \quad (66)$$

where L signifies the direction parallel to the fibers and T the transverse direction. We evaluated all quantities obtainable from the neo-classical theory for this problem and compared those results with the exact solution for various values of L/h . We have chosen to present most of our results for L/h values of 10 and 4, which represent relatively thin and thick plates, respectively.

For plotting the displacement, strain, and stress distributions the following normalized parameters are used

$$\begin{aligned} \bar{z}_\alpha &= \frac{E_T h^2 z_\alpha(x_1, \zeta)}{L^3 p_0} & \bar{z}_3 &= \frac{100 E_T h^3 z_3(x_1, \zeta)}{L^4 p_0} \\ \bar{Z}_{\alpha\beta} &= \frac{h^2 Z_{\alpha\beta}(x_1, \zeta)}{L^2 p_0} & \bar{Z}_{\alpha 3} &= \frac{h Z_{\alpha 3}(x_1, \zeta)}{L p_0} \\ \bar{\Gamma}_{ij} &= \frac{E_L h^2 \Gamma_{ij}(x_1, \zeta)}{L^2 p_0} \end{aligned} \quad (67)$$

The through-the-thickness distributions of in-plane displacement, transverse shear stress and strain are evaluated at $x_1 = 0$; while the distributions of out-of-plane displacement, in-plane stress and strain, transverse normal stress and strain are evaluated at $x_1 = L/2$. In all the results below, solid lines represent the exact solution, while dashed lines represent the present neo-classical plate theory (NCPT) results. Results from classical laminated plate theory (CPT), when distinct from NCPT results, are shown with long and short dashes.

The distributions of normalized displacements, strains, and stresses are obtained for the following three plates, in which each layer has the same thickness:

$$\begin{aligned} &[15^\circ] \\ &[15^\circ / -15^\circ] \\ &[30^\circ / -30^\circ]_{sym} \end{aligned}$$

Results for these plates are presented in Figs. 3 – 19. Figs. 3 – 5 show the transverse displacement for these three configurations, respectively. Note the very close agreement between NCPT and the exact solution. NCPT is just slightly more flexible than the exact solution; it provides significant improvement over CPT as L/h becomes small, until L/h of approximately 10.

Figs. 6 and 7 show normalized out-of-plane displacement (z_3), in-plane displacement (z_1), in-plane strain (Γ_{11}) and shear strain ($2\Gamma_{12}$) for $[15^\circ]$ plate from top to bottom in the figures; Figs. 10 and 11 show the same quantities for $[15^\circ / -15^\circ]$ plate. Figs. 8 and 9 show normalized in-plane stress (Z_{11}), transverse shear stress (Z_{13}) and strain ($2\Gamma_{13}$), transverse normal strain (Γ_{33}) for $[15^\circ]$ plate; Figs. 12 and 13 show the same quantities for $[15^\circ / -15^\circ]$ plate.

Figs. 14 – 19 show the results for $[30^\circ / -30^\circ]_{sym}$ plate. Figs. 14 and 15 show normalized out-of-plane displacement (z_3), in-plane displacements (z_1, z_2), in-plane strain (Γ_{11}). Figs. 16 and 17 show normalized in-plane shear strain ($2\Gamma_{12}$), in-plane stress (Z_{11}) and shear stress (Z_{12}), and transverse shear stress (Z_{13}). Figs. 18 and 19 show normalized transverse shear stress (Z_{23}) and strain ($2\Gamma_{23}$), transverse normal stress (Z_{33}) and strain (Γ_{33}).

Transverse in-plane stresses (Z_{22}) are not shown since these have the same pattern as the corresponding in-plane stresses (Z_{11}). Note that transverse shear stresses in the figures are obtained from integrating the equilibrium equations of elasticity. Also note that transverse in-plane displacement (z_2), in-plane shear strain and stress ($2\Gamma_{12}, Z_{12}$) and transverse shear strain and stress ($2\Gamma_{23}, Z_{23}$) will differ from those of Pagano⁷. This stems from the different sign convention used for plotting in this work, resulting in the same magnitude but with opposite sign. Some of these quantities are also reported in graphical form in Ref. 9.

The in-plane displacements and strains from NCPT shown in Figs. 6, 7, 10, 11, 14 and 15 are much closer to the exact solution than those from CPT. One reason the in-plane strain and shear strain are so accurate is that Γ_e is calculated to $O(h\varepsilon/\ell)$ using Eq. (12). The in-plane stress and shear stress are essentially identical to that of CPT shown in Figs. 8, 9, 12, and 13 for the 1- and 2-layer cases. On the other hand, NCPT is better than CPT shown in Figs. 16 and 17 for the 4-layer case. This is because that NCPT has more kinematical variables than CPT. Even though the difference is invisible for 1- and 2-layer cases, twisting curvature, which is a one of generalized strain measures used in NCPT, plays an important role for this specific example.

It is interesting to note that in Fig. 19, transverse shear stress (Z_{23}) from NCPT has sign changes around at the middle surface, while results from the exact solution are positive. Based on results obtained to date, we believe that an extension of the theory of Ref. 3 to a higher approximation still needs to be developed in order to improve the correlation of this and other three-dimensional field variables with the exact solution. However, interaction terms are present in the energy when a higher approximation is attempted, similar to those of Ref. 2; these must be killed, but the means for doing so were unknown until recently. This means was discovered in conjunction with the analysis of nonuniform thickness plates.

Higher Approximations and Plates with Nonuniform Thickness

During the last two months, we have been attempting to extend our methodology to deal with plates, the thickness of which varies spatially over the plate domain so that $h = h(x_1, x_2)$. In the process of doing this for isotropic plates, an additional degree of freedom for the normal line element was introduced on physical grounds, within the context of VAM. This degree of freedom involves the contraction of the normal line element. Although the contraction was not zero in the original theories, it was a “reactive” quantity. That is, it does not appear in the strain energy function explicitly, and it can be calculated in terms of other degrees of freedom of the normal line element. When the thickness is allowed to vary spatially over the plate, this degree of freedom appears explicitly in the reduced, two-dimensional strain energy function and must be regarded as an independent quantity. This new degree of freedom improved the correlation with three-dimensional results relative to classical theory, but the accuracy was still judged inadequate. Indeed, it was not

possible to show that the additional warping was of a higher order than this degree of freedom. Moreover, the motivation for introducing this degree of freedom was not at all systematic.

Dr. Sutyrin then discovered a powerful principle for identifying valid degrees of freedom in the dimensional reduction process. (This principle applies for development of beam, plate, and shell theories.) There exists an eigenvalue problem, the eigenvectors of which are the valid degrees of freedom for the structural member. They are valid in the sense that (1) they satisfy the appropriate interfacial conditions; (2) they guarantee that the above-mentioned interaction terms vanish; (3) they guarantee that any additional warping is of a higher order. They are not simple polynomials, in general. The degrees of freedom in classical theory, it turns out, are the eigenvectors associated with the zero eigenvalues for this eigenproblem. Sutyrin's eigen-principle should allow us to extend the earlier laminated plate theories to higher approximations in a rational manner.

If plates of constant thickness must be joined to plates of varying thickness, then both models must contain the same degrees of freedom. This means that the constant thickness model must be taken to a higher-order approximation so as to bring the same, or at least analogous degrees of freedom into the strain energies of both plates. The mathematics associated with this operation are quite difficult, involving functional analysis as well as the construction of non-trivial transformations to rid the strain energy of unneeded degrees of freedom. Unanswered questions also remain concerning the role of edge-zone behavior and penetration of the influence of self-equilibrated tractions on the plate edges.

To illustrate this, let us write the Jaumann strain components as

$$\begin{aligned}\Gamma_e &= \epsilon + \zeta Kh + \partial_e w_{||} \\ 2\Gamma_s &= w'_{||} + \partial_t w_3 \\ \Gamma_t &= w'_3\end{aligned}\tag{68}$$

The resulting warping will be different from the result of Eqs. (25), because 2γ has been set to equal to zero and its effect absorbed into w . It turns out that the warping variable can be written as

$$w_i(x_1, x_2, \zeta) = \sum_k q_k(x_1, x_2) \psi_{ik}(\zeta) + v_i(x_1, x_2, \zeta)\tag{69}$$

where v_i represents additional warping and q_i the degrees of freedom associated with the ψ functions. The ψ 's can be identified by means of the VAM procedure and Sutyrin's eigen-principle. The resulting two-dimensional strain energy function will contain the q functions and their derivatives. The process of identifying these "degrees of freedom" for the plate, as presented in the literature, is not straightforward. We have made some important progress recently in this identification procedure.

We consider a homogeneous, isotropic plate with thickness depending on only one in-plane coordinate and undergoing a stretching load; see Fig. 20. In order to solve this problem we have to introduce new degrees of freedom. It is clear that the average bending of the plate and rotating of normal line element are equal to zero here. Nevertheless there is a non-trivial deformation of normal line elements, which can be observed in the elasticity solution (which we approximated by a finite element solution). This deformation is shown schematically in Fig. 21. There is no known systematic way to introduce such deformations into classical theory or into Reissner's theory, which takes into account only the transverse shear deformation.

In order to introduce some new degrees of freedom in our analysis, we take the warping to be of the form of Eq. (69) where ψ is the shape function of a degree of freedom and q is the new degree

of freedom itself. The VAM procedure points to the best choice of the shape function ψ as an eigenvector of a natural eigenvalue problem for normal line element (a one-dimensional problem).

For example, the two degrees of freedom corresponding to the smallest nonzero eigenvalues can be represented as

$$w_\alpha = g_\alpha(x_1, x_2) \cos(2\pi\zeta) + v_\alpha \quad (70)$$

and

$$w_3 = e(x_1, x_2) \sin(\pi\zeta) + v_3 \quad (71)$$

for the homogeneous isotropic case. The degree of freedom g_1 (g for short) is responsible for in-plane deformation such as shown in Fig. 21, while the degree of freedom e describes contraction of the normal line element. The variable v models any *additional* warping, which is of a higher order and may be ignored in this example.

Fig. 22 shows comparison between g and e obtained from a 3-D solution (dashed line) and g and e from our new theory (solid line).

This approach has a natural generalization for dynamics problems. The above eigenvalues become eigenfrequencies in that context. Also, it is not difficult to obtain a suitable dynamical equation for each degree of freedom. In general, one would need to apply the theory to a wide class of problems and loading conditions in order to understand the appropriate number of degrees of freedom to be retained.

We expect to develop the dimensional reduction with the extensive use of computerized symbolic manipulation and the expertise of Dr. Sutyrin. As a fallback position, should a symbolic dimensional reduction not be possible, we can solve the minimization problems via one-dimensional finite elements (through-the-thickness). Both means are quite efficient for laminated plate problems. The through-the-thickness analysis is only done once for a given lay-up. The results for the elastic constants are used as input to the plate (two-dimensional) analysis, and the influence functions are used to recover approximations for the three-dimensional field variables once the two-dimensional problem is solved.

Future Work

In the balance of the grant we intend to apply Sutyrin's eigen-principle to develop a refined theory for laminated plates of constant thickness. If there is time we will make this theory capable of modeling the most general type of nonhomogeneous, anisotropic plate, subject only to the restrictions that the strain is small and that neither geometry nor properties vary with x_1 and x_2 . Development of an interior global deformation analysis for constant thickness anisotropic plates would then be possible, but it appears that this should be addressed only after the degrees of freedom for a general analysis are identified. This means that the boundary energy and nonuniform thickness problems should be addressed first. We are seeking to obtain funding from NASA to carry this out.

References

1. Berdichevsky, Victor L., "Variational-Asymptotic Method of Constructing a Theory of Shells," *PMM*, Vol. 43, No. 4, 1979, pp. 664 – 687.
2. Atilgan, A. R., and Hodges, D. H., "On the Strain Energy of Laminated Composite Plates," *International Journal of Solids and Structures*, 1992, to appear.

3. Hodges, D. H., Atilgan, A. R., and Danielson, D. A., "A Geometrically Nonlinear Theory of Elastic Plates," AIAA Paper 92-2281. *Proceedings of the 33rd Structures, Structural Dynamics, and Materials Conference*, Dallas, Texas, April 13 – 15, 1992, pp. 878 – 889, *Journal of Applied Mechanics*, to appear.
4. Hodges, D. H., Lee, B. W., and Atilgan, A. R., "Application of the Variational-Asymptotical Method to Laminated Composite Plates," AIAA Paper 92-2357, *Proceedings of the 33rd Structures, Structural Dynamics, and Materials Conference*, Dallas, Texas, April 13 – 15, 1992, pp. 514 – 524.
5. Reissner, E., 1972, "On Sandwich-Type Plates with Cores Capable of Supporting Moment Stresses," *Acta Mechanica*, Vol. 14, pp. 43 – 51.
6. Pagano, N. J., "Exact Solutions for Composite Laminates in Cylindrical Bending," *Journal of Composite Materials*, Vol. 3, No. 7, 1969, pp. 397 – 411.
7. Pagano, N. J., "Influence of Shear Coupling in Cylindrical Bending of Anisotropic Laminates," *Journal of Composite Materials*, Vol. 4, No. 7, 1970, pp. 330 – 343.
8. Jones, R. M., *Mechanics of Composite Materials*, McGraw Hill, New York, 1975, pp. 47 – 52.
9. Tessler, A., and Saether, E., "A Computationally Viable Higher-order Theory for Laminated Composite Plates," *International Journal for Numerical Methods in Engineering*, Vol. 31, pp. 1069 – 1086.

Undeformed State

Deformed State

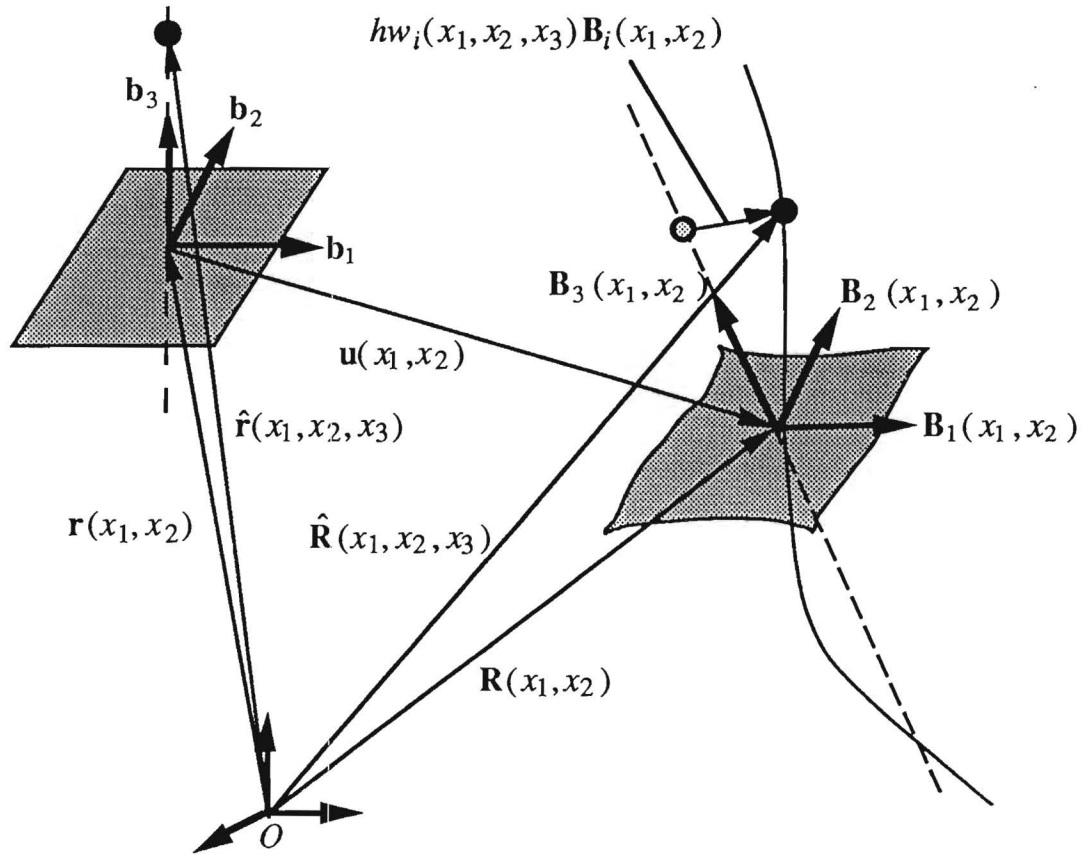


Fig. 1: Schematic of plate deformation

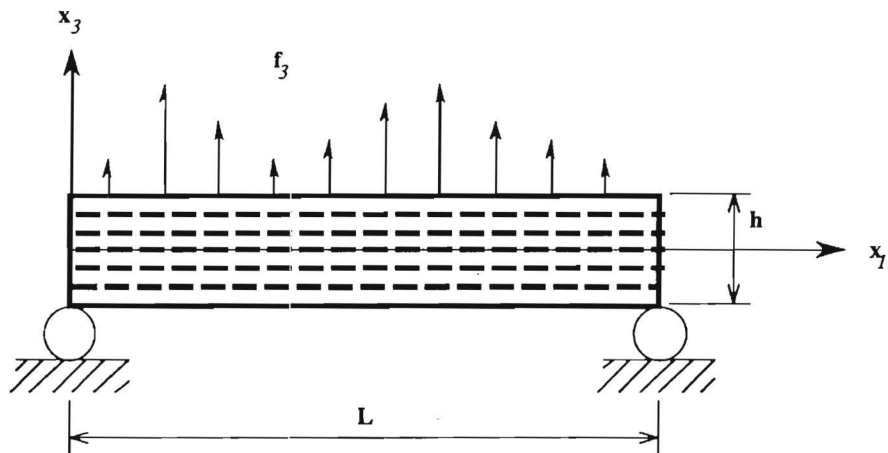


Fig. 2: Configuration of Cylindrical Bending of Plate

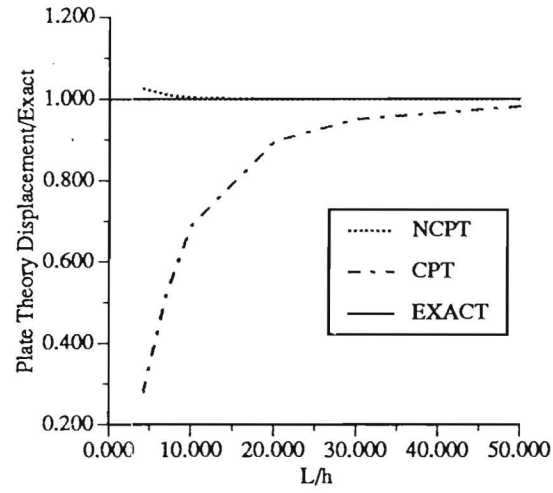


Fig. 3: Transverse Displacement for $[15^\circ]$ Plate

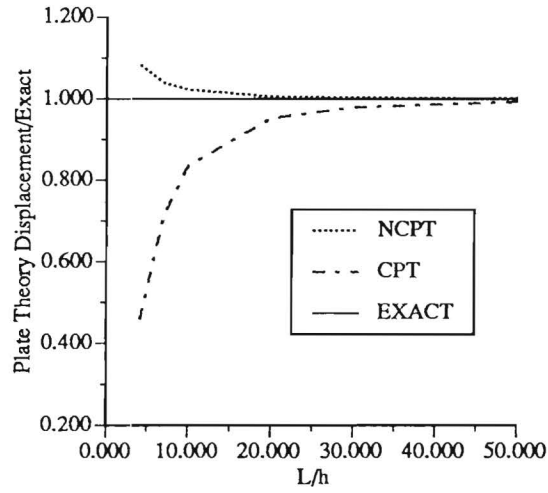


Fig. 4: Transverse Displacement for $[15^\circ / - 15^\circ]$ Plate

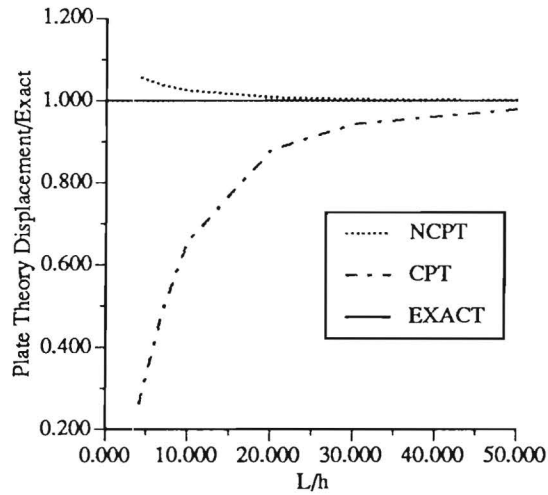
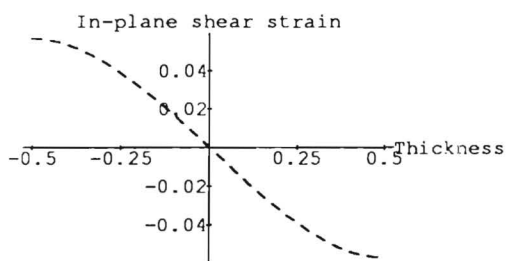
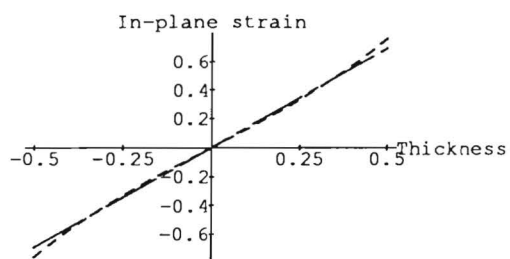
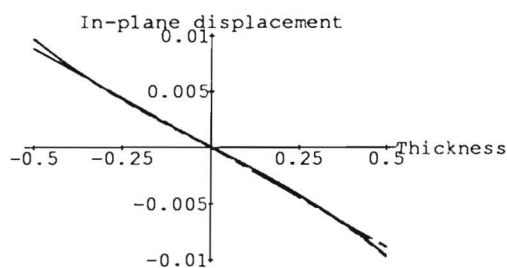
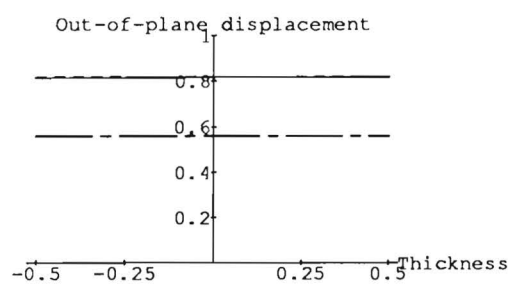
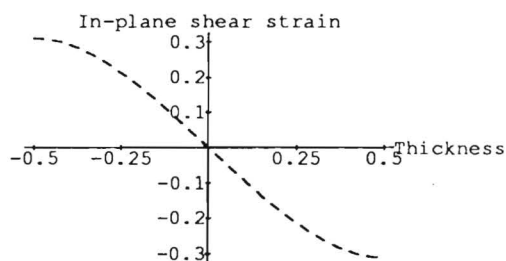
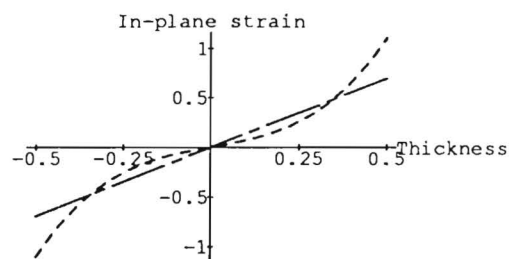
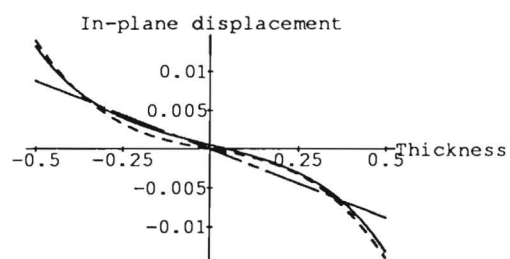
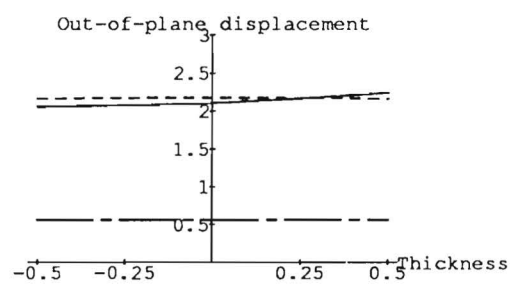


Fig. 5: Transverse Displacement for $[30^\circ / - 30^\circ]_{sym}$ Plate



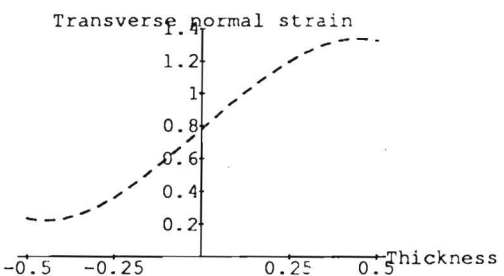
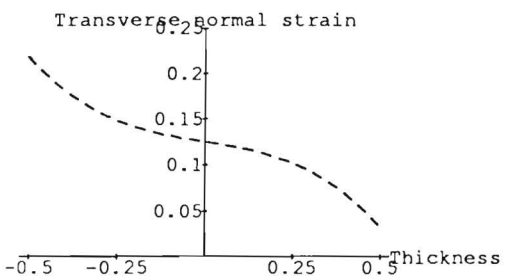
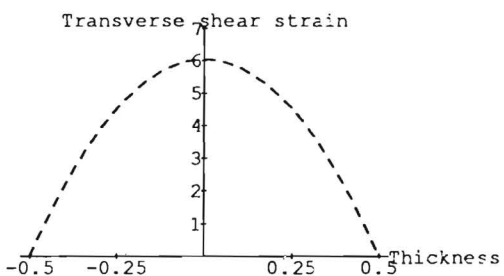
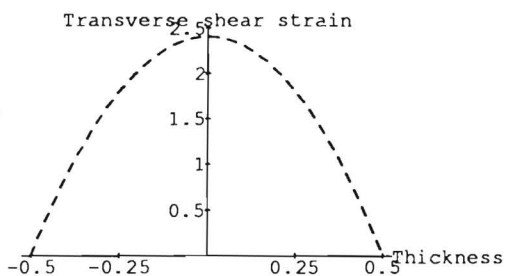
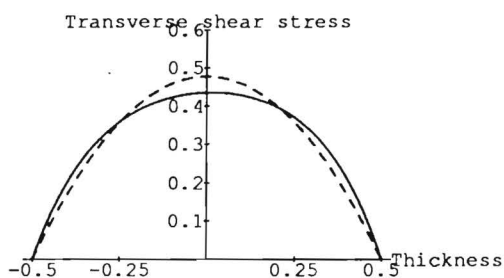
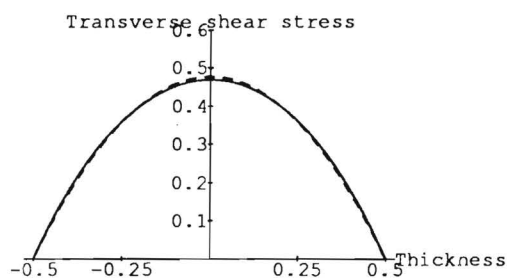
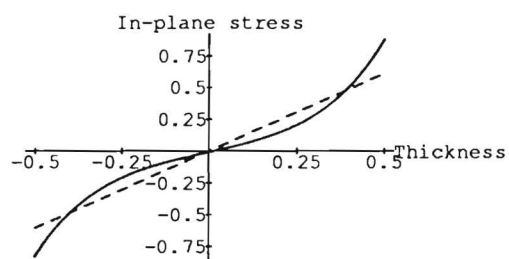
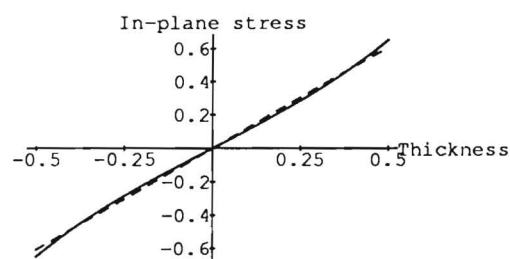
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Fig. 6: Distributions of Normalized Quantities for $[15^\circ]$ Plate when $L/h = 10$

Fig. 7: Distributions of Normalized Quantities for $[15^\circ]$ Plate when $L/h = 4$

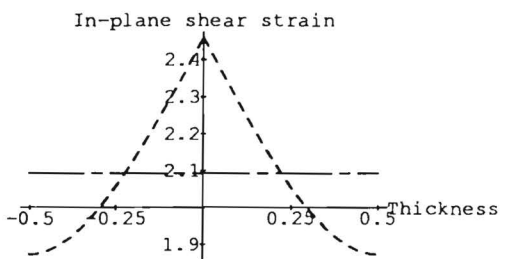
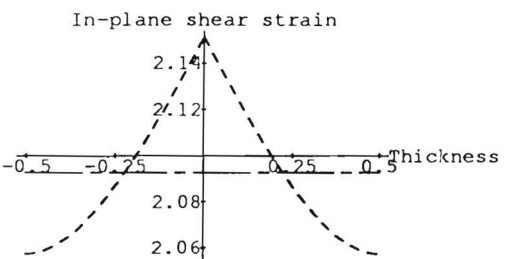
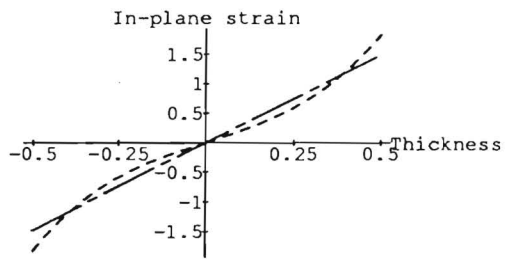
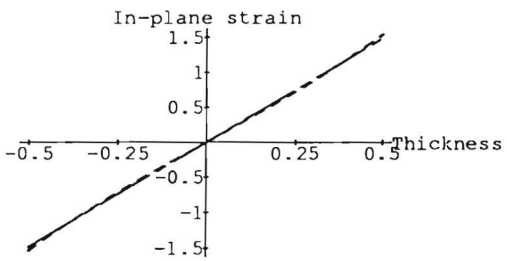
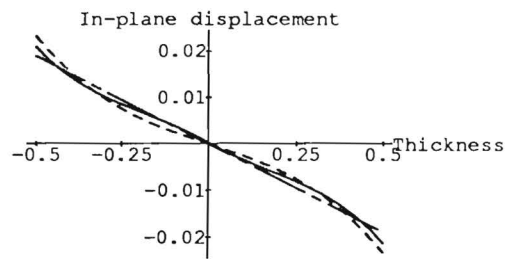
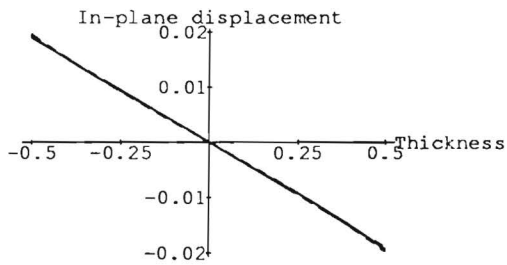
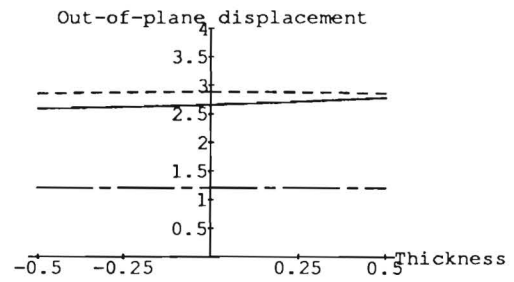
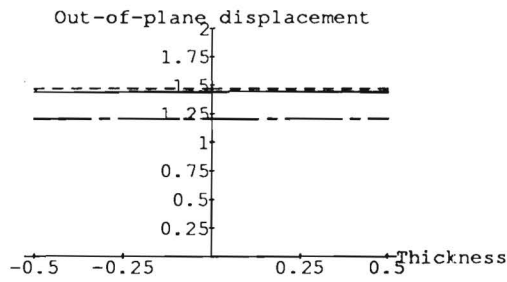


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Fig. 8: Distributions of Normalized Quantities for [15°] Plate when $L/h = 10$

Fig. 9: Distributions of Normalized Quantities for [15°] Plate when $L/h = 4$

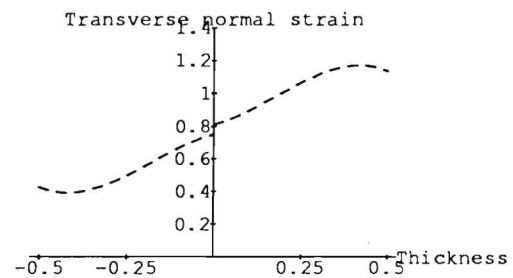
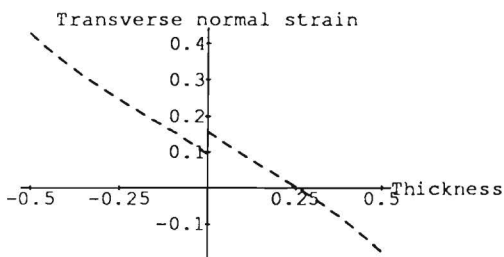
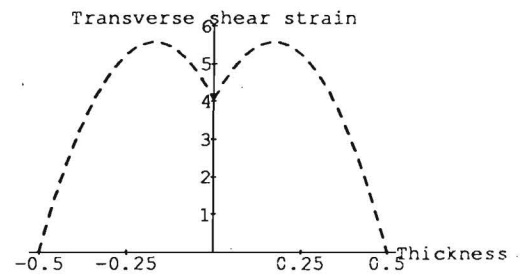
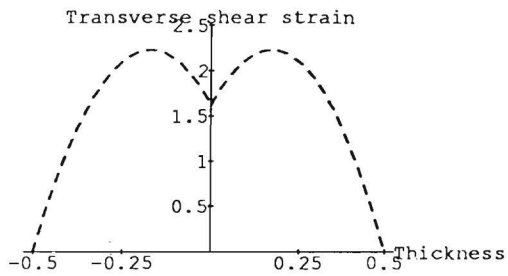
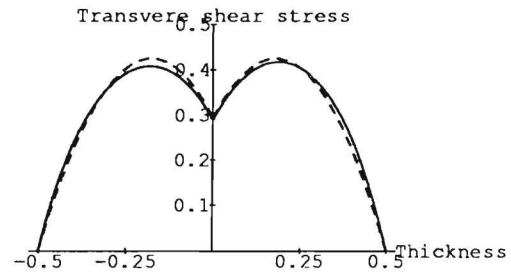
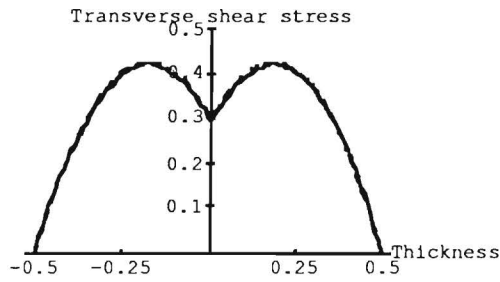
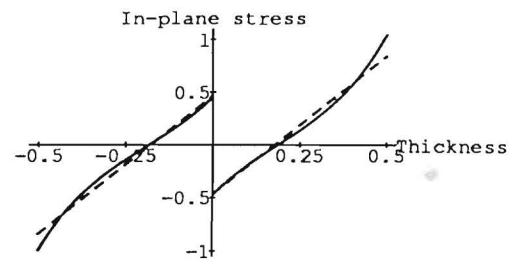
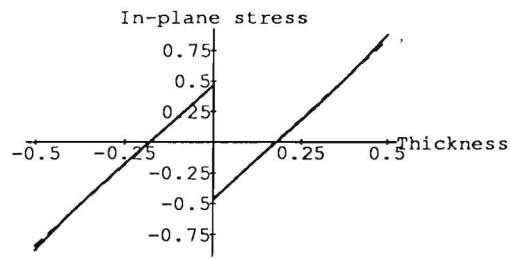


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Fig. 10: Distributions of Normalized Quantities for $[15^\circ / -15^\circ]$ Plate when $L/h = 10$

Fig. 11: Distributions of Normalized Quantities for $[15^\circ / -15^\circ]$ Plate when $L/h = 4$

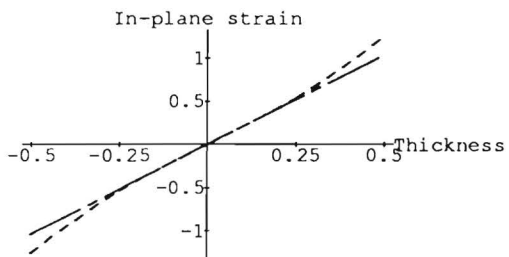
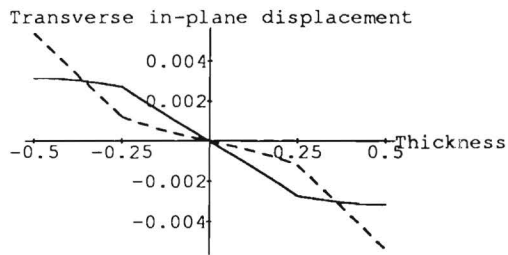
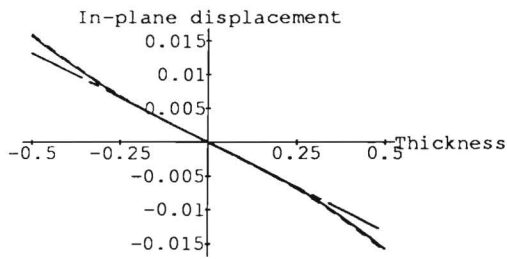
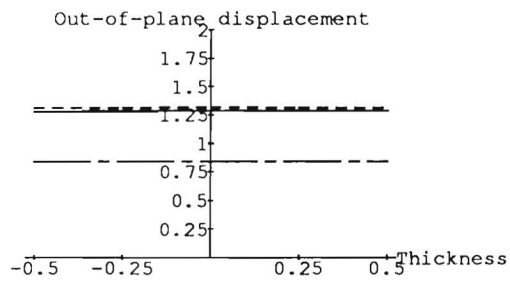


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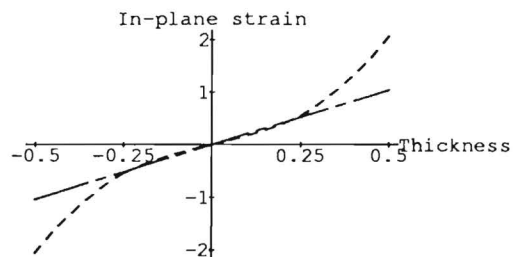
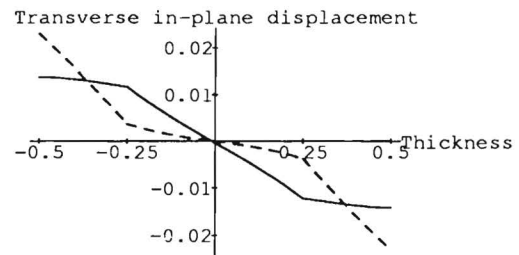
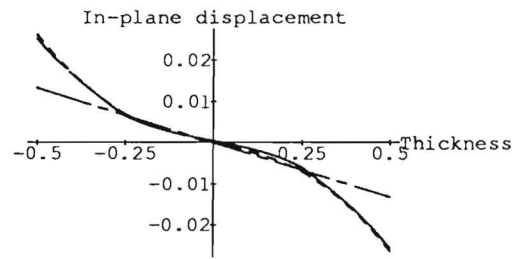
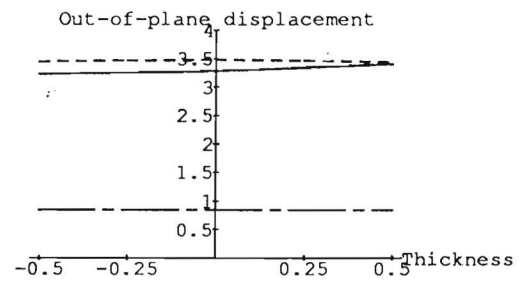
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Fig. 12: Distributions of Normalized Quantities for $[15^\circ / -15^\circ]$ Plate when $L/h = 10$

Fig. 13: Distributions of Normalized Quantities for $[15^\circ / -15^\circ]$ Plate when $L/h = 4$



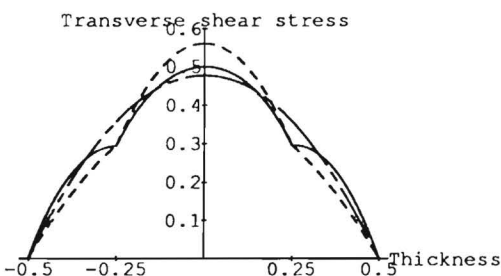
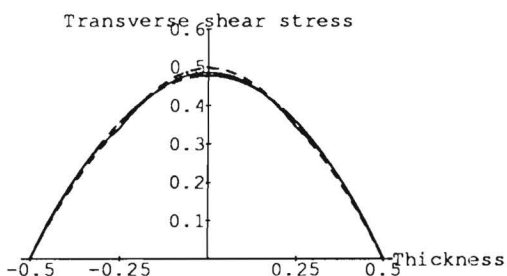
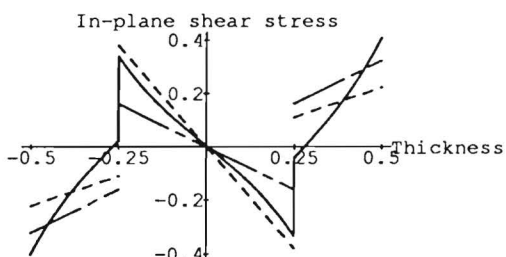
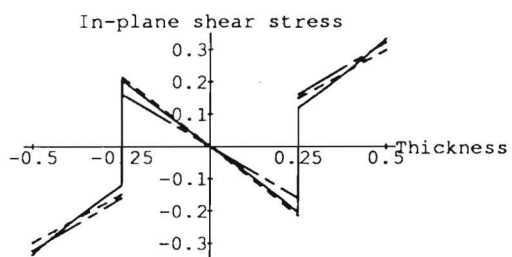
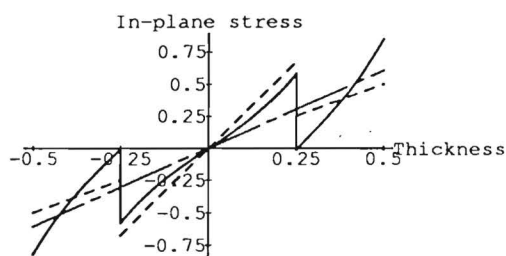
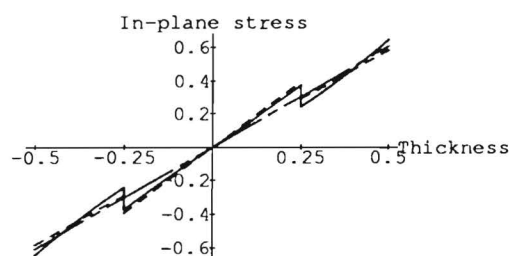
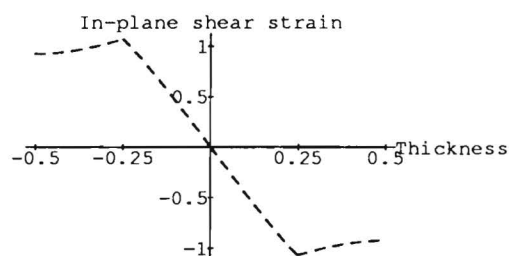
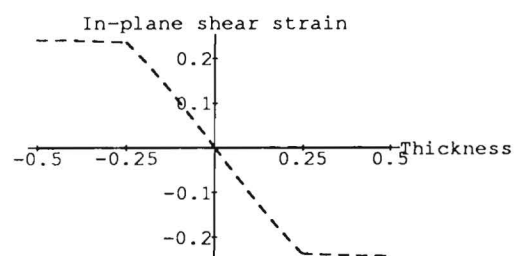
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Fig. 14: Distributions of Normalized Quantities for $[30^\circ / -30^\circ]_{sym}$ Plate when $L/h = 10$

Fig. 15: Distributions of Normalized Quantities for $[30^\circ / -30^\circ]_{sym}$ Plate when $L/h = 4$

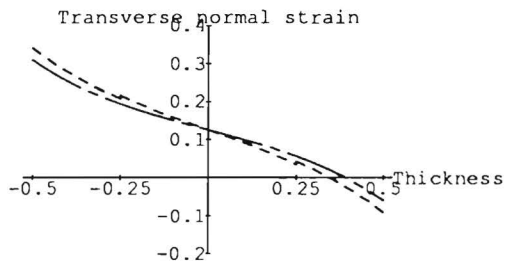
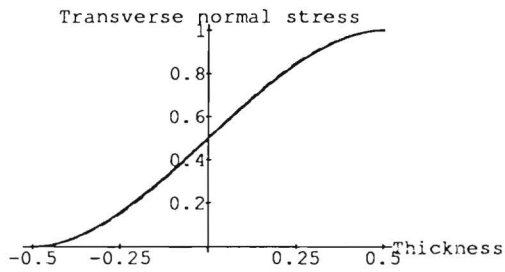
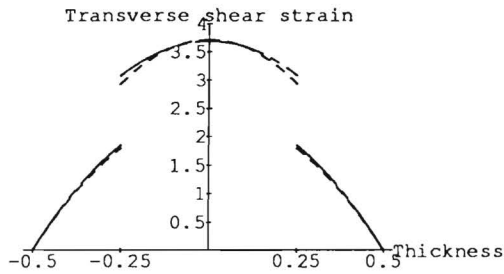
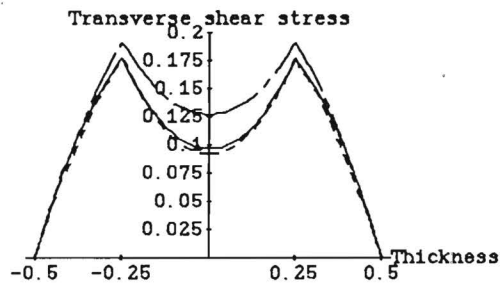


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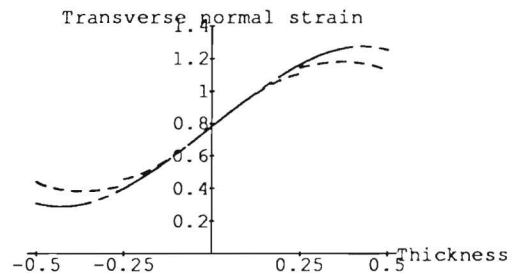
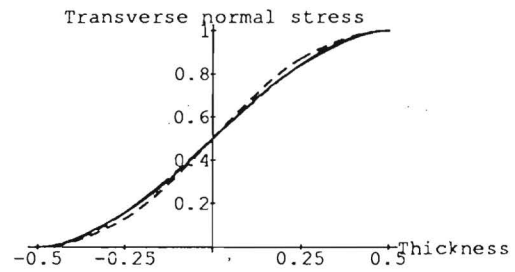
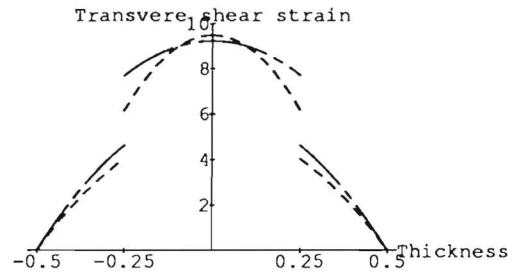
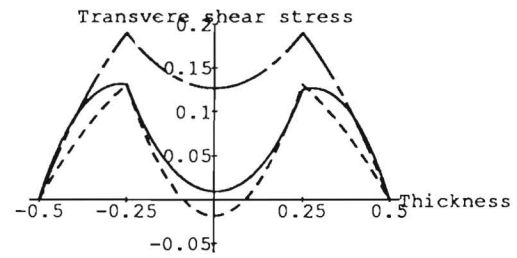
Fig. 16: Distributions of Normalized Quantities for $[30^\circ / -30^\circ]_{sym}$ Plate when $L/h = 10$

Fig. 17: Distributions of Normalized Quantities for $[30^\circ / -30^\circ]_{sym}$ Plate when $L/h = 4$



— Exact; - - - - - NCPT; - · - · - CLPT

Fig. 18: Distributions of Normalized Quantities for $[30^\circ / -30^\circ]_{sym}$ Plate when $L/h = 10$



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Fig. 19: Distributions of Normalized Quantities for $[30^\circ / -30^\circ]_{sym}$ Plate when $L/h = 4$

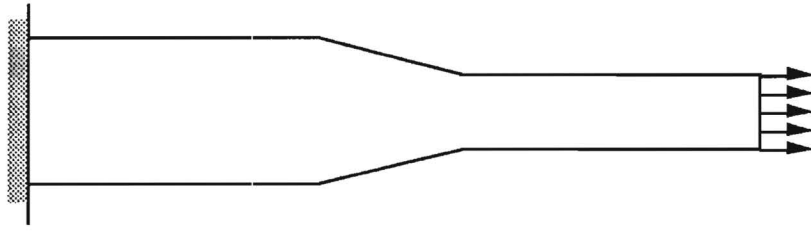


Fig. 20: Tapered plate subjected to large in-plane force

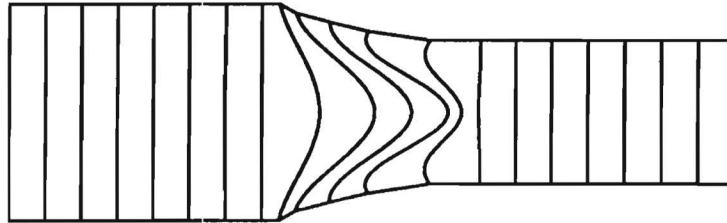


Fig. 21: Exaggerated deformation of normal line elements for tapered plate

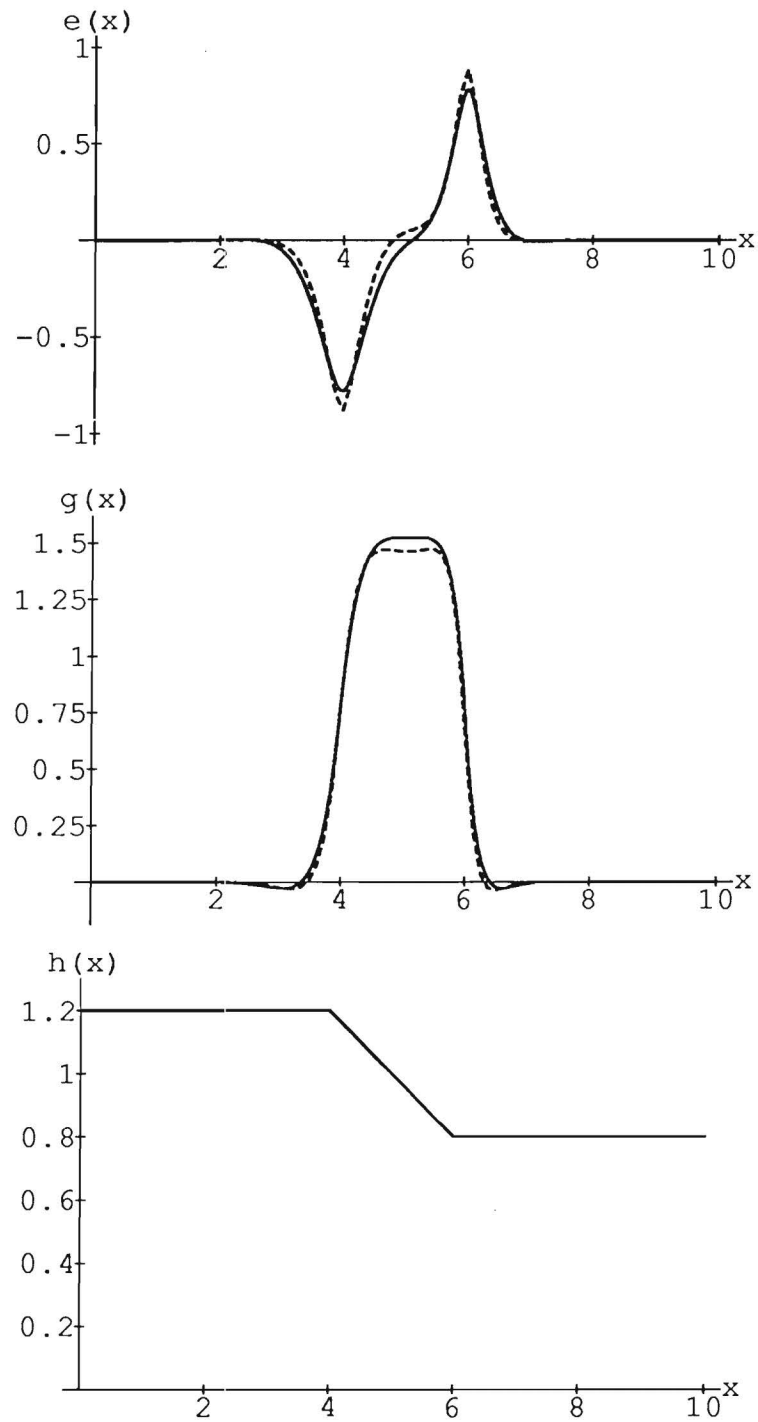


Fig. 22: Degrees of freedom from plate theory and three-dimensional results compared (solid lines are 3-D results and dashed lines are plate theory results)

Modeling of Composite Beams and Plates for Static and Dynamic Analysis

**Final Report
NASA Grant NAG-1-1094
17 Jan. 1990 – 16 Jan. 1993**

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**Research Supported by
U.S. Army Aerostructures Directorate
Technical Monitor: Mr. Howard E. Hinnant
Mail Stop 240
NASA Langley Research Center
Hampton, Virginia 23681-0001**

Summary of Work Done Under Grant Sponsorship

The main purpose of this research has been to develop a rigorous theory and corresponding computational algorithms for through-the-thickness analysis of composite plates. This type of analysis is needed in order to find the elastic stiffness constants for a plate and to post-process the resulting plate solution in order to find approximate three-dimensional displacement, strain, and stress distributions throughout the plate. This also requires the development of finite deformation plate equations which are compatible with the through-the-thickness analyses.

After about one year's work, we settled on the variational-asymptotical method¹ (VAM) as a suitable framework in which to solve these types of problems. VAM was applied to laminated plates with constant thickness in the work of Atilgan and Hodges². The corresponding geometrically nonlinear global deformation analysis of plates was developed by Hodges, Atilgan, and Danielson³. A different application of VAM, along with numerical results, was obtained by Hodges, Lee, and Atilgan⁴. An expanded version of this last paper⁵ has been submitted for publication in the *AIAA Journal*. (Copies of these papers have been delivered to Mr. Hinnant.) One last paper was just completed⁶ and a copy is included as an appendix to this report. Summaries of the progress in the various categories we worked on follow. Technical details are in the papers.

Work on Finite Deformation of Plates

In Ref. 3, a set of kinematical and intrinsic equilibrium equations are derived for large deflection and rotation but with small strain. This work showed that the drilling type rotation is not an independent variable in plate theory. If it is to be included in plate equations at all, there must be a constraint enforced between it and its definition in terms of other plate variables. Unless this constraint is enforced, the theory including the rotation as a separate variable is not valid. The main contribution of this paper is a complete set of geometrically exact strain-displacement relations for large deformation of plates.

Also, the relationship between the drilling rotation and the other kinematical variables gives new insight into the drilling moment and its role in beam-plate connectivity. An applied drilling moment at a point on a plate is not resisted at all by the plate. Such a moment, in order to have any physical resistance from the plate, must be applied over a finite area. Other than this, a point drilling moment can only be resisted by a plate if the plate model is derived from couple-stress elasticity⁷. Further study on this problem was shelved, because of the need to interpret the drilling rotational degrees of freedom in certain plate and shell finite element formulations. There is an apparent conflict between the theoretical result of a plate having zero stiffness in response to a point drilling moment on one hand, and the finite element results found in the literature for the response of plates to point drilling moments. The papers in which these results appear are of several researchers, some of whom are highly regarded by the international community and by the principal investigator. A dialog has been initiated on this subject with some of these people, but to date we have no resolution to this question.

Work on Modeling Laminated Plates

By modeling here we mean the calculation of elastic constants for a two-dimensional model based on known material constants for the three-dimensional body. This can only be done approximately, of course, and asymptotical methods are natural. In Ref. 2, the "first approximation" is exactly the same as classical laminated plate theory. The "second approximation" takes transverse shear deformation into account and is asymptotically correct only for plates with certain restrictions in their construction. To remove the restrictions one must "kill" certain interaction terms in the strain energy, but the means for doing so for general laminated plates were not given in that paper. Indeed, the means for doing this were unknown to us at that time.

The development in Refs. 4 and 5 includes transverse shear in the "first approximation" and is stopped there. Results from this theory were compared, for the cylindrical bending case, with results from the exact solution of Pagano for both cross-ply⁸ and shear-coupled⁹ laminated plates. The resulting theory, termed a "neo-classical" theory (NCPT), is at least as good as classical plate theory (CPT) in every case; for most cases NCPT is superior to CPT. For example, NCPT does a much better job on calculation of plate (two-dimensional) displacement than CPT. Also, many three-dimensional quantities are calculated much more accurately than could be achieved with CPT. Results for several different types of plates may be found in Ref. 5.

As pointed out in Ref. 5, there is a need to develop higher approximations in order to (1) improve the overall performance of the theory for applications to thick plates and (2) improve the accuracy of predicted transverse strains and stresses, especially in situations when integration of the three-dimensional equilibrium equations cannot be accomplished, such as in the geometrically nonlinear case.

Work on a Higher Approximation for NCPT

In order to improve the asymptotical accuracy of NCPT one needs to introduce "degrees of freedom" of the normal line element which will eliminate or "kill" interaction terms of the type identified in Ref. 2. In our work during the last reporting period we applied Sutyrin's eigen-principle (described in the last report) to develop a refined theory for laminated plates of constant thickness. This theory includes CPT as a special case and even in its most elementary extension should surpass NCPT in predictive capability. Furthermore, because it is based on a variable number of "degrees of freedom" for the normal line element, it should allow users of future finite elements based on this theory to decide which of these degrees of freedom they want in their models. In principle, one can approach the accuracy of three-dimensional elasticity to any degree desired. The theory is completed as far as the derivation of the stiffness model based on eigenfunctions associated with a certain Sturm-Liouville operator. A computer code has been written and validated for the calculation of these eigenfunctions. Another code is presently under development which will use these eigenfunctions to build the stiffness model. For the details of the stiffness model, see Ref. 6.

Depending on what funding we can find, in the future we intend to generate results from this new theory for the cylindrical bending problem. This will allow us to inexpensively evaluate the number and types of normal line element degrees of freedom required for accurate solutions. The last step would then be to build a prototype plate finite element. We believe that this element would be far superior to any extant plate finite elements. We would want to work with someone who already has experience in two-dimensional finite elements. The strength of our element would be in the theoretical foundation, not in the finite element technology itself. Dr. Sutyrin also has devised a means by which a discretized model based on three-dimensional finite elements can be reduced directly to plate elements. While this should be equivalent to our present semi-analytical approach, it may prove to be more convenient in certain contexts.

References

1. Berdichevsky, Victor L., "Variational-Asymptotic Method of Constructing a Theory of Shells," *PMM*, Vol. 43, No. 4, 1979, pp. 664 – 687.
2. Atilgan, A. R., and Hodges, D. H., "On the Strain Energy of Laminated Composite Plates," *International Journal of Solids and Structures*, Vol. 29, No. 20, 1992, pp. 2527 – 2543.
3. Hodges, D. H., Atilgan, A. R., and Danielson, D. A., "A Geometrically Nonlinear Theory of Elastic Plates," AIAA Paper 92-2281. *Proceedings of the 33rd Structures, Structural Dynamics, and Materials Conference*, Dallas, Texas, April 13 – 15, 1992, pp. 878 – 889. (Also, *Journal of Applied Mechanics*, Mar. 1993, to appear.)
4. Hodges, D. H., Lee, B. W., and Atilgan, A. R., "Application of the Variational-Asymptotical Method to Laminated Composite Plates," AIAA Paper 92-2357, *Proceedings of the 33rd Structures, Structural Dynamics, and Materials Conference*, Dallas, Texas, April 13 – 15, 1992, pp. 514 – 524.
5. Hodges, D. H., Lee, B. W., and Atilgan, A. R., "Application of the Variational-Asymptotical Method to Laminated Composite Plates," *AIAA Journal*, submitted, 1992.
6. Lee, B. W., Sutyrin, V. G., and Hodges, D. H., , "A Variable-Order Laminated Plate Theory Based on the Variational-Asymptotical Method," AIAA Paper No. 93-1617, *Proceedings of the 34th Structures, Structural Dynamics and Materials Conference*, La Jolla, California, April 19 – 22, 1993.
7. Reissner, E., 1972, "On Sandwich-Type Plates with Cores Capable of Supporting Moment Stresses," *Acta Mechanica*, Vol. 14, pp. 43 – 51.
8. Pagano, N. J., "Exact Solutions for Composite Laminates in Cylindrical Bending," *Journal of Composite Materials*, Vol. 3, No. 7, 1969, pp. 397 – 411.
9. Pagano, N. J., "Influence of Shear Coupling in Cylindrical Bending of Anisotropic Laminates," *Journal of Composite Materials*, Vol. 4, No. 7, 1970, pp. 330 – 343.

A Variable-Order Laminated Plate Theory Based on the Variational-Asymptotical Method

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Abstract

The variational-asymptotical method is a mathematical technique by which the three-dimensional analysis of laminated plate deformation can be split into a linear, one-dimensional, through-the-thickness analysis and a nonlinear, two-dimensional, plate analysis. The elastic constants used in the plate analysis are obtained from the through-the-thickness analysis, along with approximate, closed-form three-dimensional distributions of displacement, strain, and stress. In this paper, a theory based on this technique is developed which is capable of approximating three-dimensional elasticity to any accuracy desired. This theory is not developed using any of the usual approaches of laminated plate theory. That is, it is not based on any power series expansion through the thickness, nor is it based on introduction of a set of variables which describe displacement in separate layers of laminated plates. Rather, the asymptotical method allows for the approximation of the through-the-thickness behavior in terms of the eigenfunctions of a certain Sturm-Liouville problem associated with the thickness coordinate. These eigenfunctions contain all the necessary information about the nonhomogeneities along the thickness coordinate of the plate and thus possess the appropriate discontinuities in the derivatives of displacement. The theory is presented in this paper along with numerical results for the eigenfunctions of various laminated plates.

Introduction

For aerospace structures, laminated composite materials provide excellent opportunities for structural simplicity as well as elastic couplings for potential optimization of design criteria. Although plates made of such materials have been used for some time in a variety of engineering applications, simple and efficient methods for analyzing plates with anisotropy and nonhomogeneity are still needed. This is partly due to rapid changes taking place in manufacturing technology for composite materials and partly to ever-increasing demands for accuracy and efficiency. Much of what is done is based on classical plate theory (CPT) which,

although adequate for many engineering applications, has well known limitations due to the Kirchhoff hypothesis.

Background

Many attempts have been made to improve classical theory by taking into account non-classical behavior such as transverse shear deformation and transverse normal stresses. From the time laminated fiber-reinforced composites were first introduced, numerous works have been published, the objectives of which include the improvement of CPT for laminated plate applications. This subject is discussed at length in review papers.^{1,2} There are two main classes of methods for improving plate theory found in the literature: (1) *Power Series Methods*: expansion of the displacement field variables into higher-order power series in the thickness coordinate; and (2) *Layerwise Variables Methods*: incorporation of separate sets of displacement variables for each layer. Both of these methods have known shortcomings. For example, no power series expansion can possibly render accurate results for quantities which may possess discontinuities, such as certain components of strain and stress in laminated plates. The layerwise variables methods rely on a significant increase in the number of unknowns, a number which depends on the number of layers in the plate. A third method has received some attention in recent years, which involves an assumed displacement field with discontinuities allowed in through-the-thickness derivatives.^{3,4,5} There is no question that this method yields excellent results in some cases, but it lacks a systematic basis for choosing the displacement functions, and it does not yield an asymptotically correct result in general.

Ref. 6 undertook a quite different approach. It does not involve a power-series expansion through the plate thickness, nor does it involve layerwise unknowns. Rather, the three-dimensional energy of a laminated plate was approximated following Berdichevsky's variational-asymptotical methodology.⁷ Normally asymptotical methods are employed for analytical developments, but here such a method was used in a sort of semi-analytical approach. Namely, the theory leads to a Reissner-like plate theory, along with a set of elastic constants; it also provides a set of influence functions from which approximate three-dimensional displacement, strain, and stress fields can be determined once the plate equations are solved. The plate equations can be solved by any method desired, such as a two-dimensional finite element method. The analysis was restricted to laminated plates for which each lamina exhibits monoclinic material symmetry about its middle surface. Their first approximation is asymptotically correct for this case and coincides with classical laminated plate theory. However, their

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second approximation is asymptotically correct only when each element of the reduced stiffness matrix \bar{Q} (see Jones⁸) is constant through the thickness of the entire plate. Although the theory is not asymptotically correct otherwise, it was intended for application to laminated composite plates. The limitation of their work stems from a term in the strain energy which was neglected. A suitable method to make this term vanish rigorously is needed in order to make the theory asymptotically correct, but was not given.

A similar, but somewhat improved, approach was undertaken by Refs. 9 and 10, in which the estimation procedure of Ref. 6 was slightly modified to include transverse shear terms in the first approximation. Plates with cross-ply stacking sequences under cylindrical bending were taken as example problems in Ref. 9. In a later extension of this work,¹⁰ plates with arbitrary stacking sequences undergoing cylindrical bending were taken as example problems. The material configurations of these latter plates are not as simple as those of bi-directional plates, because of the influence of the coupling of transverse shear terms. The distributions of three-dimensional displacement, strain, and stress were investigated for both cases by comparing the corresponding three-dimensional exact elasticity solution.^{11,12} The theory of Refs. 9 and 10, termed the "neo-classical" plate theory (NCPT), was shown to be more accurate than CPT for thick, laminated plates; also, it was shown to yield results which are somewhat better than those of the theory of Ref. 6.

Still, there were results reported for which the correlation of NCPT with the exact solution is not good. For example, when shear coupling exists, NCPT shows significantly better correlation with the exact solution than for the bi-directional cases. It is necessary, then, to extend the validity of the theory to a higher approximation. Although it is not discussed in Refs. 9 and 10, such an extension requires that certain interaction terms vanish. These terms are analogous to the one neglected in Ref. 6. For this reason a method for generalization of the theory of Refs. 9 and 10 has been developed, and that is the subject of the present paper.

Present Approach

The essence of the new approach, which guarantees disappearance of the interaction terms discussed above, is the introduction of new "degrees of freedom" into the three-dimensional displacement field. We are not using the term "degrees of freedom" in its usual sense. Here we mean plate (i.e., two-dimensional) displacement variables which are associated with a particular deformation mode of the normal line element through the thickness. The shape functions of these new degrees of freedom are chosen as the eigenfunctions associated with a certain Sturm-Liouville problem formulated by Sutyrin.¹³ By choosing the associated warping displacement to be orthogonal to the shape functions for each of the new degrees of freedom, the displacement field is uniquely defined. The choice of shape functions from these

eigenfunctions guarantees that any additional warping induced by the new degrees of freedom is of a higher order relative to the new degrees of freedom themselves. The order of these new degrees of freedom relative to the strain depends on the loading and the material constants. After obtaining the eigenvectors by using a one-dimensional finite element method, the plate elastic constants can be obtained through the variational-asymptotical method. Utilizing global deformation equations along with this resulting plate constitutive law, we will complete the formulation of the theory.

In this paper we first provide the details for the kinematics of plate deformation in terms of classical plate displacement variables and the new degrees of freedom. The strain field needed to develop a geometrically nonlinear plate theory is written in terms of these displacement variables. The small parameters are identified as ϵ , the maximum strain in the plate, and $\frac{h}{\ell}$, where h is the plate thickness and ℓ is the characteristic length over which the deformation varies in the deformed plate. The variational-asymptotical method is then used, along with a Ritz-type approximation of three-dimensional displacement variables in the through-the-thickness coordinate, to approximate the three-dimensional strain energy of the plate with a function of two-dimensional quantities only. The above-mentioned Sturm-Liouville problem is identified, the eigenfunctions of which guarantee the warping to be of higher order than any retained degree of freedom if they are chosen as the shape functions for the retained degrees of freedom. The two-dimensional strain energy function is then given as a function of material constants and eigenfunctions.

We are able to calculate the exact solution for the Sturm-Liouville problem only for one- and two-layer plates. Thus, it was necessary to develop an approximate method of solution so that the plate theory could be finalized. We present a finite element analysis in one dimension (through the thickness) which we use to solve the Sturm-Liouville problem based on the shape functions of Ref. 14. Eigenvalues and eigenfunctions for various laminated plates obtained by a one-dimensional finite element method are presented and, when possible, compared with the exact solution. Application of the theory will be addressed in a later paper.

Theoretical Development

The objective is to derive a strain energy function of a plate in terms of two-dimensional quantities only. It can be done only if some small parameters are present. We suppose that the parameters mentioned above, ϵ and $\frac{h}{\ell}$, are small.

Three-Dimensional Formulation

To begin we will formulate a three-dimensional development which shall be considered the exact solution to the plate problem.

Undeformed State of Plate A typical point in the undeformed plate can be located by introducing a Cartesian coordinate system x_i in such a way that $x_\alpha: \{x_1, x_2\} \equiv x$ denotes

lengths along orthogonal straight lines in the mid-surface of the undeformed plate, and $x_3 \equiv y \equiv h\zeta$ is the distance in the normal direction, where $-\frac{1}{2} \leq \zeta \leq \frac{1}{2}$. Throughout the analysis, Greek indices assume values 1 or 2; Latin indices assume values 1, 2, and 3; and repeated indices are summed over their ranges.

The spatial position vector $\hat{r}(x_1, x_2, y)$ to an arbitrary point in the undeformed plate can be written as

$$\hat{r}(x, y) = r(x) + yb(x) \quad (1)$$

where $r(x)$ is the spatial position vector of points on the mid-surface of a plate and $b(x)$ is the unit normal vector. We will also need notation for unit vectors $b_\alpha \equiv \frac{\partial r}{\partial x_\alpha}$ which, together with b , form an orthonormal triad.

Since the variable y (and ζ) is chosen specifically so that the spatial position vector r to a point on the reference mid-surface is the average position of points along the normal line at a particular value of x_1 and x_2 , then

$$r(x) = \langle \hat{r}(x, \zeta) \rangle \quad (2)$$

where the notation

$$\langle \bullet \rangle \equiv \frac{1}{h} \int_{-\frac{1}{2}}^{\frac{1}{2}} \bullet dy \equiv \int_{-\frac{1}{2}}^{\frac{1}{2}} \bullet d\zeta \quad (3)$$

is used throughout the paper.

Deformed State of Plate Without any restrictions the position vector $\hat{R}(x_1, x_2, y)$ to an arbitrary point in the deformed plate can be represented by

$$\hat{R}(x, y) = R(x) + yB(x) + v_n(x, \zeta)B_n(x) \quad (4)$$

where $R(x)$ is the position vector of points on the deformed reference surface, $B_n(x)$: $\{B_1(x), B_2(x), B_3(x) \equiv B(x)\}$ is the reference orthonormal triad with vector B being orthogonal to the deformed reference surface, and $v_n(x, \zeta)$ are components of the general warping displacement of an arbitrary point in the deformed normal line, consisting of both in- and out-of-plane components so that all possible deformations are considered.

The warping v_n could not be defined uniquely as a function of ζ with an arbitrary choice of $R(x)$ unless they are subject to the constraints

$$\langle v_n(x, \zeta) \rangle = 0 \quad (5)$$

which means that

$$R(x) = \langle \hat{R}(x, \zeta) \rangle \quad (6)$$

The orthogonality of vector B to the reference surface means

$$R_{,\alpha} \cdot B = 0 \quad (7)$$

where the notation $(\cdot)_{,\alpha}$ denotes the partial derivative with respect to x_α .

In order to eliminate the arbitrary rotation of vectors B_α around normal B we impose the following constraint

$$B_1 \cdot R_{,2} = B_2 \cdot R_{,1} \quad (8)$$

A schematic of the plate deformation is shown in Fig. 1.

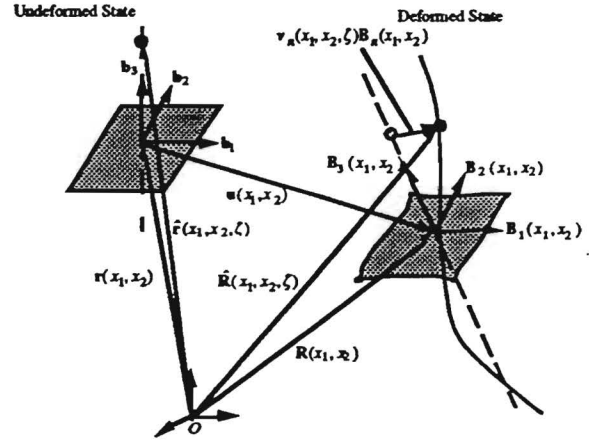


Fig. 1: Schematic of plate deformation (u is the displacement of the reference surface)

Thus, Eq. (4) provides a convenient way to represent the arbitrary function $\hat{R}(x, y)$. The orientation of the triad B_n is now specified uniquely.

Strain Field As shown in Ref. 6, under the condition of small local rotation, the Jaumann strain components can be arranged in a 3×3 symmetric matrix Γ^* , given by

$$\Gamma^* = \frac{1}{2}(\chi + \chi^T) - I \quad (9)$$

$$\chi_{ij} = B_i \cdot \frac{\partial \hat{R}}{\partial x_j}$$

where I is the 3×3 identity matrix.

Substituting Eq. (4) into the Jaumann strain, Eq. (9), one can express the strain field as a 6×1 column matrix $\Gamma = [\Gamma_{11}^* \ 2\Gamma_{12}^* \ \Gamma_{22}^* \ 2\Gamma_{13}^* \ 2\Gamma_{23}^* \ \Gamma_{33}^*]^T$ so that

$$\Gamma = \frac{1}{h} \Gamma_h v' + \Gamma_\epsilon \gamma + \Gamma_l v_{,x} \quad (10)$$

where $(\bullet)' \equiv \frac{\partial \bullet}{\partial \zeta}$ and matrices Γ_h , Γ_ℓ and Γ_ℓ are

$$\Gamma_h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Gamma_\ell = \begin{bmatrix} I & \zeta I \\ 0 & 0 \end{bmatrix} \quad (11)$$

$$\Gamma_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Gamma_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\gamma = \begin{Bmatrix} \epsilon \\ hK \end{Bmatrix} \quad v_x = \begin{Bmatrix} v_{,1} \\ v_{,2} \end{Bmatrix} \quad (12)$$

Here we will introduce the two-dimensional strain measures for inplane stretching and shear

$$\epsilon = \begin{Bmatrix} R_{,1} \cdot B_1 - 1 \\ R_{,1} \cdot B_2 + R_{,2} \cdot B_1 \\ R_{,2} \cdot B_2 - 1 \end{Bmatrix} \quad (13)$$

and the two-dimensional strain measures for bending and twist

$$K = \begin{Bmatrix} B_{,1} \cdot B_1 \\ B_{,1} \cdot B_2 + B_{,2} \cdot B_1 \\ B_{,2} \cdot B_2 \end{Bmatrix} \quad (14)$$

The small parameter ϵ can be now specified as

$$\epsilon = \max \|\gamma\| \quad (15)$$

A few nonlinear terms in the strain field, which couple v and γ , have been neglected in Eq. (10) because the physically linear plate theory is considered. The form of the strain field is of great importance because it is now linear in γ , v' , and $v_{,z}$.

This is the only point where ϵ as a small parameter is taken into account.

Strain Energy of Plate The strain energy density per unit area for a plate can be written as

$$U = \frac{1}{2} \langle \Gamma^T D \Gamma \rangle \quad (16)$$

where D is the 6×6 symmetric material matrix in the B_n basis.

The three-dimensional Jaumann stress Z , which is conjugate to the Jaumann strain Γ is

$$Z = D\Gamma \quad (17)$$

Basic Three-Dimensional Problem

The basic three-dimensional problem can now be represented as the following minimization problem

$$\int U(\gamma(x), v'(x, \zeta), v_{,x}(x, \zeta)) \, h dx_1 dx_2 + \text{terms with external forces} \rightarrow \min \quad (18)$$

where the minimum should be found with respect to three arbitrary functions $R(x)$, through which $\gamma(x)$ is calculated, and to three functions $v_n(x, \zeta)$ which are subject to the constraints of Eq. (5).

Note that the orthonormal triad B_n is not an independent variable, since it is a subject to the constraints in Eqs. (7) and (8). It is completely determined by the function $R(x)$.

Dimensional Reduction

Splitting of the Problem Now the functional space of all the admitted functions $\tilde{R}(x, y)$ is split into sublayers with a choice of the x -dependent functions $R(x)$. In each layer the functions $v_n(x, \zeta)$ are arbitrary under the constraint in Eq. (5).

We can solve this problem in two steps. First we are going to find functions $v_n(x, \zeta)$ for any prescribed choice of functions $R(x)$. As a result, we will have $v_n(x, \zeta)$ as a functional of $R(x)$ and ζ , and the functional, Eq. (18), will become dependent only on $R(x)$. That functional will give us a two-dimensional plate theory. The second step is to solve that theory.

Since the energy density U depends not only on functions $v_n(x, \zeta)$ but also on their derivatives with respect to x , then the result of the first step will be very complicated (it will contain a non-local dependence on $R(x)$ in the general case) and cannot be obtained in an appropriate form unless we take advantage of some small parameters.

Small Parameters Let us consider the situation where parameters h , ℓ and ϵ are present. Since no coefficient matrix of Γ , Eq. (11), depends on these parameters, it is clear that the first term of the expression for strain, Eq. (10), has order $\frac{\|v\|}{h}$, the second term has order ϵ , and the third one has order $\frac{\|v\|}{\ell}$. The third term has order $\frac{h}{\ell}$ times that of the first. We should neglect this as a higher order term in the first approximation if we are going to expand the solution with respect to the small parameter $\frac{h}{\ell}$. In the future, this important circumstance allows us to avoid the presence of derivatives of unknown functions $v_n(x, \zeta)$ with respect to x in the problem for any approximation and, hence, to solve it in an appropriate form.

Since our main problem has become linear with respect to the unknown functions $v_n(x, \zeta)$ and the two-dimensional strain measure γ , the smallness of parameter ϵ does not

need to be considered any more. This fact has already taken into account (see above under *Strain Field*). We will expand the warping $v_n(x, \zeta)$ as a series with respect to the small parameter $\frac{h}{l}$ only. In the absence of other small parameters, expansion in $\frac{h}{l}$ is the same as expansion in h . That is why we can consider h to be the small parameter in spite of its dimension.

Discretization

The problem may be solved numerically by discretizing with respect to the thickness variable ζ . Now the unknown functions $v_n(x, \zeta)$ are represented as the product of a matrix of shape functions $S(\zeta)$ and a column matrix of nodal values of $v(x, \zeta)$, which we will denote V

$$v(x, \zeta) = S(\zeta)V(x) \quad (19)$$

Substituting the discretized unknown function in Eq. (19) into the energy density, Eq. (16), while taking into account the strain, Eq. (10), one obtains

$$2U = \left(\frac{1}{h}\right)^2 V^T E V + \left(\frac{1}{h}\right) 2 V^T [D_{he} \gamma + D_{hx} V_{,x}] + \quad (20)$$

$$+ (1) (\gamma^T D_{ee} \gamma + 2 V_{,x}^T D_{xe} \gamma + V_{,x}^T D_{xx} V_{,x})$$

in which the following definitions are introduced

$$\begin{aligned} E &\triangleq \langle [\Gamma_h S]^T D [\Gamma_h S] \rangle \\ D_{he} &\triangleq \langle [\Gamma_h S]^T D [\Gamma_e] \rangle \\ D_{hx} &\triangleq \langle [\Gamma_h S]^T D [\Gamma_l S] \rangle \\ D_{ee} &\triangleq \langle [\Gamma_e]^T D [\Gamma_e] \rangle \\ D_{xe} &\triangleq \langle [\Gamma_l S]^T D [\Gamma_e] \rangle \\ D_{xx} &\triangleq \langle [\Gamma_l S]^T D [\Gamma_l S] \rangle \end{aligned} \quad (21)$$

Classical Considerations

According to the variational-asymptotical procedure, in order to get the next approximation, one should retain only the leading energy term with respect to the small parameter that contains the unknown functions and the leading intersection term between the unknown function and the rest of the functional (for more details see Ref. 7).

We are left with the following expression

$$\left(\frac{1}{h}\right)^2 V^T E V + \left(\frac{1}{h}\right) 2 V^T D_{he} \gamma \quad (22)$$

This function should be minimized with respect to variable V under the constraint, Eq. (5), which is transformed to the following condition after discretization

$$V^T H \Psi_{cl} = 0 \quad H \triangleq \langle S^T S \rangle \quad (23)$$

where Ψ_{cl} is matrix with three columns, each corresponding to one of the constraints of Eq. (5). The set of columns Ψ_{cl} are determined by the kernel (null-space) of matrix E (for more details see Ref. 7). This means

$$E \Psi_{cl} = 0 \quad (24)$$

Let us suppose that the set of columns Ψ_{cl} is normalized in such a way that

$$\Psi_{cl}^T H \Psi_{cl} = I \quad (25)$$

The Euler equation for the problem posed by Eqs. (22) and (23) is

$$\left(\frac{1}{h}\right) E V + D_{he} \gamma = H \Psi_{cl} \mu \quad (26)$$

where μ is the column matrix of Lagrange multipliers for the constraint in Eq. (23). By pre-multiplying Eq. (26) by Ψ_{cl}^T one can prove that

$$\mu = \Psi_{cl}^T D_{he} \gamma \quad (27)$$

Now the equation for V , Eq. (26), is rewritten as

$$\left(\frac{1}{h}\right) E V = -(I - H \Psi_{cl} \Psi_{cl}^T) D_{he} \gamma \quad (28)$$

Since E has a kernel, E^{-1} does not exist. However, the pseudo-inverse of E , E_{cl}^+ , satisfied the following relations

$$\begin{aligned} E E_{cl}^+ &= I - H \Psi_{cl} \Psi_{cl}^T \\ E_{cl}^+ E &= I - \Psi_{cl} \Psi_{cl}^T H \\ E_{cl}^+ E E_{cl}^+ &= E_{cl}^+ \end{aligned} \quad (29)$$

can be found (see the Appendix) and the solution of Eq. (28) can be represented as

$$V = -h E_{cl}^+ D_{he} \gamma \quad (30)$$

Substituting the solution, Eq. (30), into the discretized strain energy density, Eq. (20), and keeping only terms with the lowest order, which are equal to $h^0 \equiv 1$ one obtains

$$2U = \gamma^T A \gamma \quad (31)$$

with

$$A \triangleq D_{ee} - [D_{he}]^T E_{cl}^+ [D_{he}] \quad (32)$$

The third property, Eq. (29), is taken into account here.

This is identical to the classical result for the strain energy of laminated plates.

New Degrees of Freedom

In order to make our plate functional more flexible with respect to the variable x , let us introduce the new unknown plate functions such that

$$V(x) = \Psi_q q(x) + W(x) \quad (33)$$

where q is a column matrix of several new unknown functions, Ψ_q is a matrix, each column of which represents a ζ -shape form associated with one of the new unknown functions $q(x)$, which will be named "new degrees of freedom," and W is the new warping to be found.

We will suppose that matrix Ψ_q is normalized in such a way that

$$\Psi_q^T H \Psi_q = I \quad (34)$$

The following constraint for W will make the splitting, Eq. (33), unique

$$W^T H \Psi_q = 0 \quad (35)$$

The order of functions $q(x)$ with respect to h may be arbitrary and it will be supposed to be equal to $h^0 \equiv 1$.

As a ζ -shape form of new degrees of freedom let us take eigenvectors of matrix E which correspond to the several lowest non-zero eigenvalues. Such a matrix Ψ_q will satisfy the following equation

$$E \Psi_q = H \Psi_q \Lambda_q \quad (36)$$

where Λ_q is a diagonal matrix of eigenvalues

$$\Lambda_q = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{N_q} \end{bmatrix} \quad (37)$$

The constraint of Eq. (23), which still might be satisfied by W , can be combined with the constraint of Eq. (35) after introducing the matrix $\Psi_u = [\Psi_\alpha \Psi_q]$ in such a way that

$$W^T H \Psi_u = 0 \quad (38)$$

Analogously, Eq. (36) can be rewritten as

$$E \Psi_u = H \Psi_u \Lambda_u \quad (39)$$

where matrix Λ_u is

$$\Lambda_u = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \lambda_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \lambda_{N_q} \end{bmatrix} \quad (40)$$

Calculation of Strain Energy

Let us assume that we have the correct expansion of V through order h^2

$$V = V_0 + h V_1 + h^2 V_2 \quad (41)$$

where V_0 denotes the first term of Eq. (33)

$$V_0 = \Psi_q q \quad (42)$$

The vector V_0 satisfies the equation

$$E V_0 = H \Psi_q \Lambda_q q \quad (43)$$

and vectors V_1 and V_2 have to satisfy the constraint found in Eq. (38).

If we have an asymptotically correct expansion for Eq. (41), we can calculate an asymptotically correct energy for order $h^0 \equiv 1$

$$\begin{aligned} 2U = & \left(\frac{1}{h}\right)^2 V_0^T E V_0 + \\ & + \left(\frac{1}{h}\right) 2 V_0^T [E V_1 + D_{h\epsilon} \gamma + D_{hx} V_{0,x}] + \\ & + (1) \{ V_1^T E V_1 + 2 \underline{V_2^T E V_0} + \\ & + 2 V_1^T [D_{h\epsilon} \gamma + D_{hx} V_{0,x}] + 2 V_{1,x}^T D_{xx} V_0 + \\ & + \gamma^T D_{\epsilon\epsilon} \gamma + 2 V_{0,x}^T D_{x\epsilon} \gamma + V_{0,x}^T D_{xx} V_{0,x} \} \end{aligned} \quad (44)$$

The underlined terms are equal to zero here because of Eq. (43) for V_0 and the constraint of Eq. (38) for V_1 and V_2 . This means we do not need to know the second approximation for V (i.e., V_2) in order to calculate the energy for order h^0 .

We shall minimize the functional

$$V_1^T E V_1 + 2 V_1^T [D_{h\epsilon} \gamma + (D_{hx} - D_{xh}^*) V_{0,x}] \quad (45)$$

in order to find V_1 .

The notation D_{xh}^* means

$$D_{xh}^* = [(D_{hx1})^T \quad (D_{hx2})^T] \quad (46)$$

which comes from $2 V_{1,x}^T D_{xx} V_0$ in the fourth line of Eq. (44) after integration by parts with respect to x_1 and x_2 .

The Euler equation for the functional of Eq. (45) is

$$EV_1 + D_{he}\gamma + (D_{hx} - D_{xh}^*)V_{0,x} = H\Psi_u\mu_u \quad (47)$$

where μ_u is the Lagrange multiplier which enforces the constraint in Eq. (38).

Applying a procedure similar to the one used for the classical case we can calculate the Lagrange multiplier μ_u

$$\mu_u = \Psi_u^T [D_{he}\gamma + (D_{hx} - D_{xh}^*)V_{0,x}] \quad (48)$$

and represent the solution of Eq. (47) by

$$V_1 = E_u^+ [D_{he}\gamma + (D_{hx} - D_{xh}^*)V_{0,x}] \quad (49)$$

where the matrix E_u^+ can be found with the following properties

$$\begin{aligned} EE_u^+ &= I - H\Psi_u\Psi_u^T \\ E_u^+E &= I - \Psi_u\Psi_u^TH \\ E_u^+EE_u^- &= E_u^+ \end{aligned} \quad (50)$$

See the Appendix for an explanation of how to calculate the matrix E_u^+ .

Substituting this expression into the energy, Eq. (44), one obtains

$$\begin{aligned} 2U &= \left(\frac{1}{h}\right)^2 V_0^T EV_0 + \left(\frac{1}{h}\right) 2 V_0^T [D_{he}\gamma + D_{hx}V_{0,x}] + \\ &+ (1)\{\gamma^T A_{ee}\gamma + 2 V_{0,x}^T P_{xe}\gamma + V_{0,x}^T P_{xx}V_{0,x}\} \end{aligned} \quad (51)$$

where the following notations are introduced

$$\begin{aligned} A_{ee} &\triangleq D_{ee} - [D_{he}]^T E_u^+ [D_{he}] \\ P_{xe} &\triangleq D_{xe} - [D_{hx} - D_{xh}^*]^T E_u^+ [D_{he}] \\ P_{xx} &\triangleq D_{xx} - [D_{hx} - D_{xh}^*]^T E_u^+ [D_{hx} - D_{xh}^*] \end{aligned} \quad (52)$$

Finally, after substituting the expression for V_0 , Eq. (42), the strain energy can be written as

$$2U = \begin{Bmatrix} \gamma \\ \frac{1}{h}q \\ q,x \end{Bmatrix}^T \begin{bmatrix} A_{ee} & A_{eq} & A_{ex} \\ A_{qe} & A_{qq} & A_{qx} \\ A_{xe} & A_{xq} & A_{xx} \end{bmatrix} \begin{Bmatrix} \gamma \\ \frac{1}{h}q \\ q,x \end{Bmatrix} \quad (53)$$

where

$$\begin{aligned} A_{qq} &\triangleq \Psi_q^T E \Psi_q = A_q \\ A_{qx} &\triangleq [\Psi_q^T D_{hx_1} \Psi_q \quad \Psi_q^T D_{hx_2} \Psi_q] = A_{qx}^T \\ A_{qe} &\triangleq \Psi_q^T D_{he} = A_{eq}^T \\ A_{xx} &\triangleq \begin{bmatrix} \Psi_q^T P_{x_1 x_1} \Psi_q & \Psi_q^T P_{x_1 x_2} \Psi_q \\ \Psi_q^T P_{x_2 x_1} \Psi_q & \Psi_q^T P_{x_2 x_2} \Psi_q \end{bmatrix} \\ A_{ex} &\triangleq [P_{ex_1} \Psi_q \quad P_{ex_2} \Psi_q] = A_{xe}^T \end{aligned} \quad (54)$$

Eq. (53) represents the strain energy of a laminated plate undergoing deformation which is constrained only in the sense that the strain is small. Displacement and rotation of the normal line element appear nonlinearly in the expressions for γ . On the other hand, the new degrees of freedom give rise to simple linear Euler equations. Since the displacement field is now completely specified in terms of γ , q , and q,x , it becomes a simple matter to recover strain and stress throughout the plate by use of Eqs. (10) and (17). The classical plate energy can be obtained from Eq. (53) after it has been minimized with respect to variables q with partial derivatives q,x being equal to zero.

Numerical Results

Exact solutions for the Sturm-Liouville problem can be obtained with symbolic manipulation software, but we were only able to carry this out for one- and two-layer plates. Thus, we turned to a finite element solution. In Ref. 15 results for Sturm-Liouville problems with discontinuous coefficients were obtained which agree quite well with the exact solution. Our finite element method is similar except that the orthogonal Jacobi-polynomial-based shape functions presented in Ref. 14 are used. This means that our finite elements have interior degrees of freedom which can be added without generation a new element geometry. This mesh through the thickness can be as fine as we wish, but we must at least have element breaks where discontinuities exist. The highest derivatives involved in the problem are first derivatives with respect to ζ , and thus C^0 continuous shape functions can be used. See Ref. 14 for details.

In this section we present some numerical results for the solutions of the Sturm-Liouville problem, compared when possible with the exact solution. We start with an error analysis of the eigenvalues for the two-layer case, and we conclude with the eigenfunctions of one-, two-, and four-layer plates. The four-layer results include both a symmetric plate and a non-symmetric cross-ply plate.

For the purpose of discussion only, we align our coordinate axes with x_1 along the length ("longitudinal inplane") and x_2 along the width ("lateral inplane"); these are not necessarily aligned with any material axes. For our examples, we choose a fiber reinforced composite material which has the following material properties

$$\begin{aligned} E_L &= 25 \times 10^6 \text{ psi} & E_T &= 10^6 \text{ psi} \\ G_{LT} &= 0.5 \times 10^6 \text{ psi} & G_{TT} &= 0.2 \times 10^6 \text{ psi} \\ \nu_{LT} &= \nu_{TT} &= 0.25 \end{aligned}$$

where signifies the direction parallel to the fibers and T the transverse direction. These properties, along with a ply angle, allow the calculation of the matrix D . Note, however, that in the example problems and indeed in all laminated plates, certain terms in D will vanish. Our theory does not require any special terms in D to be zero. Because of the vanishing terms in the example problems, there is no coupling between out-of-plane and inplane displacement components. Thus, certain modes will be entirely inplane

and others entirely out-of-plane. Also, since D_{66} is constant through the thickness for such plates, the thickness mode with the lowest eigenvalue for all the examples will be a sine function.

As examples we consider the following four lay-ups:

$[15^\circ]$	"one-layer"
$[-15^\circ/15^\circ]$	"two-layer"
$[30^\circ/-30^\circ]_{sym}$	"four-layer"
$[0^\circ/90^\circ/0^\circ/90^\circ]$	"cross-ply"

where the words in quotation marks indicate the terms we use to designate the plate under consideration.

Eigenvalues for Two-Layer Plate

All of the plates have three zero eigenvalues. These correspond with the classical "degrees of freedom" of the normal line element, the average displacement components of that line element. The eigenvalues appear in the above equations as well, and the size of these eigenvalues may have a bearing on whether the associated degree of freedom is an important one. Our present understanding is that the smallest eigenvalues are the ones of interest, but this must be confirmed by application of the theory with different choices for the degrees of freedom. Here we will present a few of the smallest non-zero eigenvalues for the two-layer plate and the analysis of the error, since the exact solution is available for this case.

In Table 1 the first four nonzero eigenvalues are shown for the two-layer plate, both from the exact solution and from our finite element approximation with four elements ($M=4$). The order of the shape functions is varied by changing J , the number of Jacobi polynomials used to construct the higher-order shape functions. The crudest element has $J=0$, resulting in linear shape functions. Results are shown for $J=1$ (quadratic shape functions) and $J=2$ (cubic shape functions). It is seen that all of the finite element results are very close to the exact solution. Furthermore, the higher-order shape functions greatly improve the accuracy.

Exact	Numerical Results ($M=4$)		
	$J=0$	$J=1$	$J=2$
2.04599617	2.15317042	2.04710246	2.04600108
4.29843523	4.54270883	4.30275294	4.29845836
8.87418665	10.84944490	8.93773416	8.87567158
10.57173730	11.12555748	10.57715155	10.57176142

Table 1: Eigenvalues for 2-layer plate

A more precise analysis of the error for the lowest nonzero eigenvalue is presented in Figs. 2 and 3. In Fig. 2 the relative error versus the size of the eigenvalue problem is essentially a straight line on a log-log scale if J , the number of Jacobi polynomials in each element, is held constant. This means that the size of the matrix increases as the number of elements through the thickness is increased. In Fig. 3

the relative error versus the size of the eigenvalue problem is shown for the case when the mesh is held constant; here the size of the matrix increases as the order of element shape functions is increased. We will present our results below for the eigenfunctions based on elements with cubic shape functions. To analyze plates with a large number of layers, the number of elements must be increased. If only the higher-order functions were used, such analyses could become expensive. However, as seen in these plots, an increase in the number of elements, which would be required because of discontinuities in the properties between layers, would bring the error down without the need of higher-order shape functions.

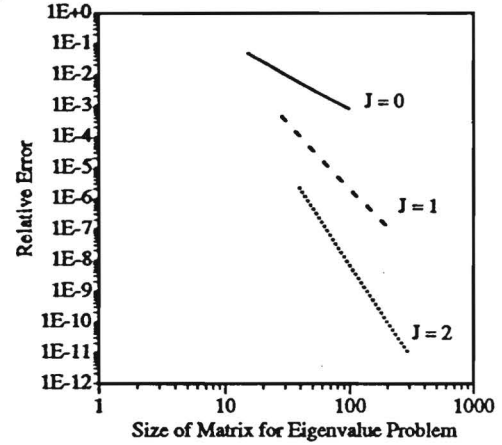


Fig. 2: Relative error for λ_1 of 2-layer plate versus size of matrix E for constant J

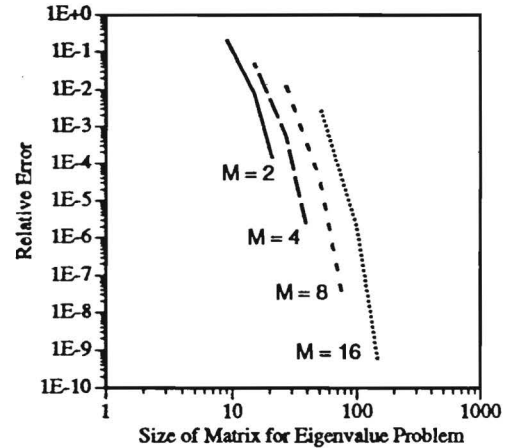


Fig. 3: Relative error for λ_1 of 2-layer plate versus size of matrix E for constant M

Eigenfunctions for One-Layer Plate

Eigenfunctions for the smallest four nonzero eigenvalues of the one-layer plate are presented in Figs. 4a - 4d. The finite element results, shown as solid lines, were obtained using 2 elements with cubic shape functions; and the exact solution, shown as dashed lines, is indistinguishable

from the finite element results in the plots. Fig. 4a shows the eigenfunction for a transverse shear mode dominated by the lateral inplane displacement, while Fig. 4b shows the corresponding one dominated by the longitudinal inplane displacement. Note in Fig. 4a the two displacement components are of opposite sign while in Fig. 4b they are of the same sign. Fig. 4c shows a higher mode of the same type as shown in Fig. 4a. The fourth nonzero eigenvalue has an eigenfunction with only out-of-plane displacement. We note the smooth character of these functions, which are close to sines and cosines. Plate theories based on expansion of the displacement in power series or trigonometric should provide excellent predictive capability for homogeneous plates such as this one.

Eigenfunctions for Two-Layer Plate

Eigenfunctions for the smallest four nonzero eigenvalues of the two-layer plate are presented in Figs. 5a – 5d, along with the corresponding eigenvalues. The finite element results were obtained using 4 elements with cubic shape functions, and the exact solution is again indistinguishable from the finite element results in the plots. Fig. 5a shows the eigenfunction for a transverse shear mode dominated by the lateral inplane displacement. Fig. 5b shows the corresponding mode having more longitudinal inplane displacement, but not so strongly dominated by it. Fig. 5c shows a higher mode of the same type as shown in Fig. 5a. The fourth nonzero eigenvalue again has an eigenfunction with only out-of-plane displacement, as shown in Fig. 5d.

Although the displacement functions shown in Figs. 5c and 5d both appear to be smooth, the displacement has a discontinuous slope in both Figs. 5a and 5b, in sharp contrast to results for the one-layer plate. Power series approximations through the entire thickness of the plate would not be able to capture this behavior, but the above theory shows that the warping induced by this type of displacement is a higher-order effect. Thus, a plate theory with these degrees of freedom should be a noticeable improvement over classical theory. Furthermore, such a theory should be an improvement over the theory of Refs. 9 and 10.

Eigenfunctions for Four-Layer Plate

Eigenfunctions for the smallest four nonzero eigenvalues of the four-layer plate are presented in Figs. 6a – 6d. These results were obtained using 8 elements with cubic shape functions. Figs. 6a and 6b show the eigenfunctions for coupled transverse shear modes, the former having more lateral inplane displacement while the latter has more longitudinal inplane displacement. Fig. 6c shows a higher mode of the same type as shown in Fig. 6a. The fourth nonzero eigenvalue again has a smooth eigenfunction with only out-of-plane displacement, shown in Fig. 6d, as expected. Again, in sharp contrast to results for the one-layer plate, the displacement has discontinuous slopes in Figs. 6a – 6c. The symmetry of the plate is exhibited in the symmetry of the modes.

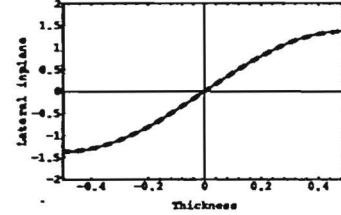
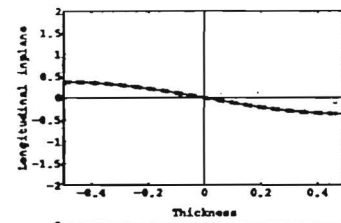


Fig. 4a: $\lambda = 1.97392088$

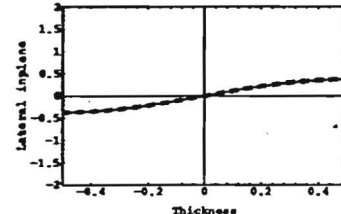
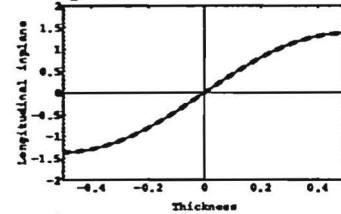


Fig. 4b: $\lambda = 4.9348022$

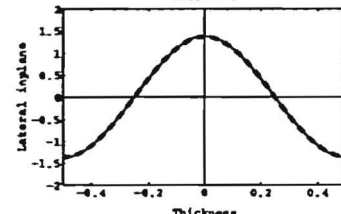
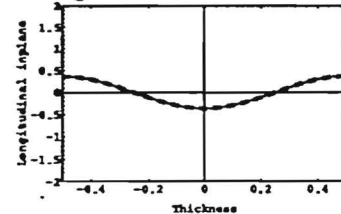


Fig. 4c: $\lambda = 7.89568352$

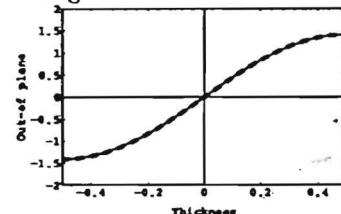


Fig. 4d: $\lambda = 10.57173733$

Fig. 4: Eigenfunctions for 1-layer plate

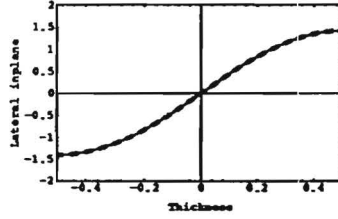
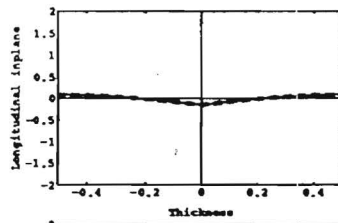


Fig. 5a: $\lambda = 2.04599617$

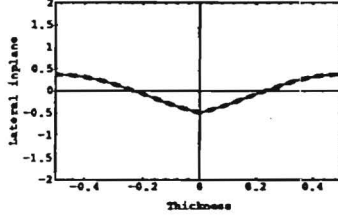
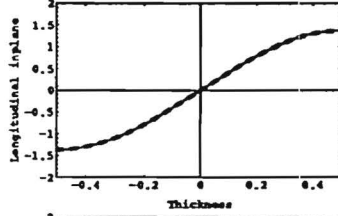


Fig. 5b: $\lambda = 4.29843523$

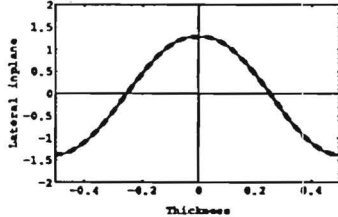
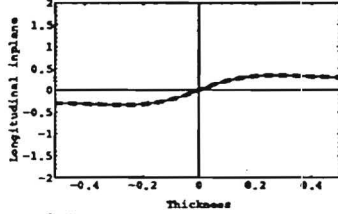


Fig. 5c: $\lambda = 8.87418665$

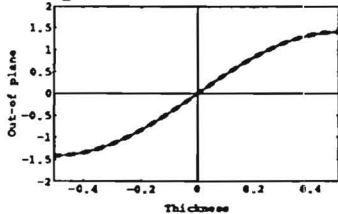


Fig. 5d: $\lambda = 10.57173730$

Fig. 5: Eigenfunctions for 2-layer plate

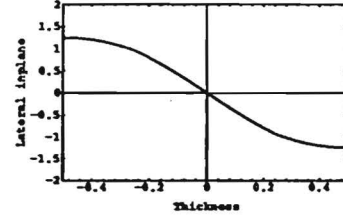
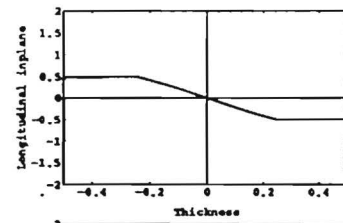


Fig. 6a: $\lambda = 2.12930862$

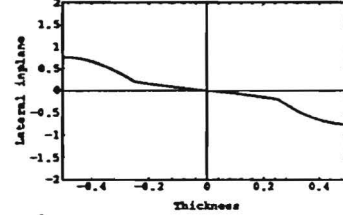
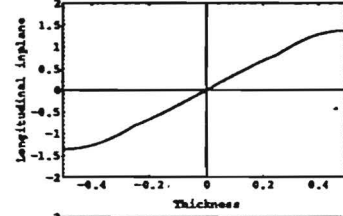


Fig. 6b: $\lambda = 4.06979407$

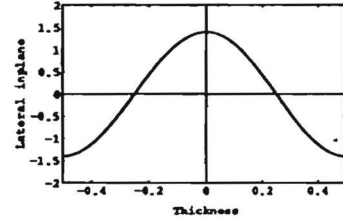
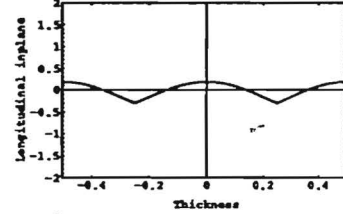


Fig. 6c: $\lambda = 9.10254004$

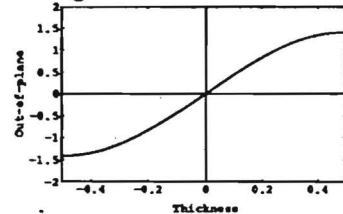


Fig. 6d: $\lambda = 10.57173771$

Fig. 6: Eigenfunctions for 4-layer plate

Eigenfunctions for Cross-Ply Plate

Eigenfunctions for the smallest four nonzero eigenvalues of the cross-ply plate are presented in Figs. 7a – 7d. These results were obtained using 8 elements with cubic shape functions. Figs. 7a and 7b show the eigenfunctions for coupled transverse shear modes (with the same eigenvalues), one having more lateral inplane displacement while the other has more longitudinal inplane displacement. The third, fourth, and fifth nonzero eigenvalues are also the same. The eigenfunctions corresponding to two of these eigenvalues are shown in Figs. 7c and 7d. The out-of-plane mode is shown in Fig. 7c, having the same smooth eigenfunction as before. Fig. 7d shows a higher mode of the same type as shown in Fig. 7b. The other eigenfunction for this triple root (not shown) is a higher mode of the type as shown in Fig. 7a. Again, in contrast to results for the one-layer plate, the displacement has discontinuous slopes – this time in Figs. 7a, 7b, and 7d. The lack of symmetry of the plate is exhibited in a lack of symmetry in the inplane modes.

It should be clear that choosing *a priori* displacement fields which exhibit the character of the inplane modes shown in Figs. 4 – 7 would be virtually impossible. They have the appropriate symmetry or asymmetry, as well as the discontinuities which reflect the layup.

Concluding Remarks

A geometrically nonlinear theory for laminated plates is presented based on a combination of the variational-asymptotical method and the method of Ritz. The displacement field is described in terms of the average displacement of the normal line element and a small number of additional functions of the in-plane coordinates of the plate. The through-the-thickness shape functions for these new "degrees of freedom" are not analytic functions for arbitrarily laminated plates. Rather, they are eigenfunctions of a certain Sturm-Liouville problem based on the thickness coordinate of the plate. Unlike power series formulations, this allows for the correct treatment the known jumps in the stress and strain fields. Unlike layerwise variable theories, the present theory has only a small number of variables in addition to those found in classical plate theory, a number which does not depend on the number of layers in the plates. Additional equilibrium equations for the plate theory associated with the new degrees of freedom are simple, linear equations – even for a large-displacement theory.

Since analytical solutions of the Sturm-Liouville problem are limited to one- and two-layer plates, an approximate finite element solution was obtained. Results obtained for these shape functions are presented for a variety of laminated plate configurations. These results agree well with available exact solutions. In the future numerical studies will be conducted in order to determine how many and which types of degrees of freedom produce the best all-around plate theory.

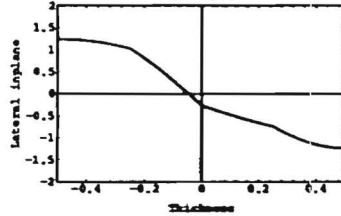
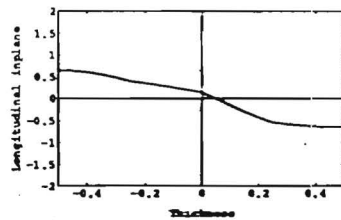


Fig. 7a: $\lambda = 2.78541496$

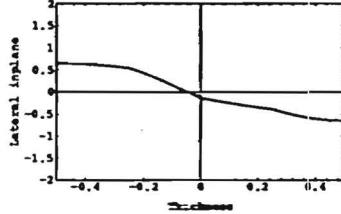
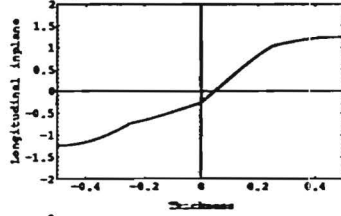


Fig. 7b: $\lambda = 2.78541496$

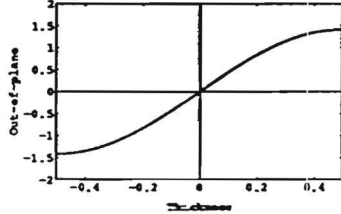


Fig. 7c: $\lambda = 10.57176142$

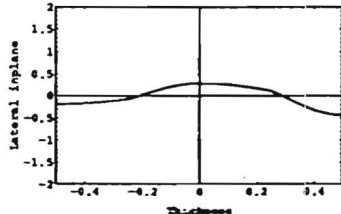
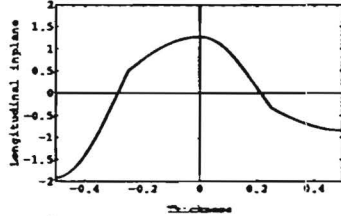


Fig. 7d: $\lambda = 10.57176142$

Fig. 7: Eigenfunctions for cross-ply plate

Acknowledgements

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References

- Noor, Ahmed K., and Burton, Scott W., "Assessment of Shear Deformation Theories for Multilayered Composite Plates," *Applied Mechanics Reviews*, Vol. 41, No. 1, 1989, pp. 1 - 13.
- Librescu, L., and Reddy, J. N., "A Few Remarks Concerning Several Refined Theories of Anisotropic Composite Laminated Plates," *International Journal of Engineering Science*, Vol. 27, No. 5, 1989, pp. 515 - 527.
- Murakami, H., "Laminated Composite Plate Theory with Improved In-Plane Responses," *J. Applied Mech.*, vol. 53, 1986, pp. 661 - 666.
- Di Sciuva, M., "Bending, Vibration and Buckling of Simply Supported Thick Multilayered Orthotropic Plates: An Evaluation of a New Displacement Model," *J. Sound Vib.*, vol. 105, no. 3, 1986, pp. 425 - 442.
- Cho, M., "An Efficient Higher-Order Composite Plate Theory for General Lamination Configurations," *Proceedings of the 32nd Structures, Structural Dynamics and Materials Conference*, Baltimore, Maryland, April 8 - 10, 1991, pp. 1496 - 1510 (AIAA Paper 91-1204).
- Atilgan, A. R., and Hodges, D. H., "On the Strain Energy of Laminated Composite Plates," *International Journal of Solids and Structures*, Vol. 29, No. 20, 1992, pp. 2527 - 2543.
- Berdichevsky, Victor L., "Variational-Asymptotic Method of Constructing a Theory of Shells," *PMM*, Vol. 43, No. 4, 1979, pp. 664 - 687.
- Jones, R. M., *Mechanics of Composite Materials*, McGraw Hill, New York, 1975, pp. 47 - 52.
- Hodges, D. H., Lee, B. W., and Atilgan, A. R., "Application of the Variational-Asymptotical Method to Laminated Composite Plates," AIAA Paper 92-2357, *Proceedings of the 33rd Structures, Structural Dynamics, and Materials Conference*, Dallas, Texas, April 13 - 15, 1992, pp. 514 - 524.
- Hodges, D. H., Lee, B. W., and Atilgan, A. R., "Application of the Variational-Asymptotical Method to Laminated Composite Plates," *AIAA Journal*, submitted, 1992.
- Pagano, N. J., "Exact Solutions for Composite Laminates in Cylindrical Bending," *Journal of Composite Materials*, Vol. 3, No. 7, 1969, pp. 397 - 411.
- Pagano, N. J., "Influence of Shear Coupling in Cylindrical Bending of Anisotropic Laminates," *Journal of Composite Materials*, Vol. 4, No. 7, 1970, pp. 330 - 343.
- Sutyrin, V. G., "Some Problems of Averaging Micro-nonhomogeneous Periodic Media," (in Russian) *Ph.D. Thesis*, Department of Mathematics and Mechanics, Moscow University, 1984.
- Hodges, D. H., "Orthogonal Polynomials as Variable-Order Finite Element Shape Functions," *AIAA Journal*, vol. 21, no. 5, May 1983, pp. 796 - 797.
- Hodges, D. H., "Direct Solutions for Sturm-Liouville Systems with Discontinuous Coefficients," *AIAA Journal*, vol. 17, no. 8, Aug. 1979, pp. 924 - 926.

Appendix

Calculation of Pseudo-Inverse Matrix

Let columns of matrix Ψ be the set of all eigenvectors. In other words the matrix Ψ satisfies the following system of equations

$$E\Psi = H\Psi\Lambda$$

$$\Psi^T H \Psi = I$$

where the diagonal matrix of eigenvalues Λ is

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \lambda_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \lambda_{N_q} & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \lambda_{N_q+1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & \lambda_{N-3} \end{bmatrix}$$

where N is the dimension of matrix E .

Now, it is easy to check that Eq. (50) and the following expressions are true

$$E = H\Psi\Lambda\Psi^T H$$

$$\Psi\Psi^T = H^{-1}$$

$$E_r^{-1} = \Psi\Lambda_r^{-1}\Psi^T$$

where Λ_r^{-1} is

$$\Lambda_r^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{\lambda_{N_q+1}} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & \frac{1}{\lambda_{N-3}} \end{bmatrix}$$